ON THE PROFILE OF SIGN CHANGING SOLUTIONS OF AN ALMOST CRITICAL PROBLEM IN THE BALL

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Abstract. We study the existence and the profile of sign-changing solutions to the slightly subcritical problem

$$-\Delta u = |u|^{2^*-2-\varepsilon}u \text{ in } \mathcal{B}, \quad u = 0 \text{ on } \partial\mathcal{B},$$

where $\mathcal{B}$ is the unit ball in $\mathbb{R}^N$, $N \geq 3$, $2^* = \frac{2N}{N-2}$ and $\varepsilon > 0$ is a small parameter. Using a Lyapunov-Schmidt reduction we discover two new non-radial solutions having 3 *bubbles* with different nodal structures. An interesting feature is that the solutions are obtained as a local minimum and a local saddle point of a reduced function, hence they do not have a global min-max description.

Mathematics Subject Classification 2010: 35B33, 35B40, 35B44, 35J20, 35J61, 35J91

Keywords: slightly subcritical problem, sign-changing solutions, blow-up, finite-dimensional reduction, nodal regions

1. Introduction and main result

The paper is concerned with the slightly subcritical elliptic problem

$$-\Delta u = |u|^{2^*-2-\varepsilon}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.1}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^N$, $N \geq 3$, $\varepsilon > 0$ is a small parameter. Here 2^* denotes the critical exponent in the Sobolev embeddings, i.e. $2^* = \frac{2N}{N-2}$. This problem has received a lot of attention, in particular with respect to investigating the lack of compactness of the critical problem where $\varepsilon = 0$. Whereas most papers deal with positive solutions and their blow-up behavior as $\varepsilon \to 0$ we deal with sign-changing solutions.

In [25] Pohožaev proved that the problem (1.1) does not admit a nontrivial solution if Ω is star-shaped and $\varepsilon \leq 0$. On the other hand problem (1.1) has a positive solution if $\varepsilon \leq 0$ and Ω is an annulus, see Kazdan and Warner [21]. In [3] Bahri and Coron found a positive solution to (1.1) with $\varepsilon = 0$ provided that the domain Ω has a nontrivial topology. The slightly supercritical case $\varepsilon < 0$ was studied in [14, 15, 16, 23] where the authors proved solvability of (1.1) for $\varepsilon < 0$ sufficiently small in a domain with one or more small holes and found positive solutions which blow up at 2 or more points of the domain as ε goes to zero, i. e. appropriately scaled solutions u_ε converge, as $\varepsilon \to 0^+$, towards a sum of delta distributions.

In the subcritical case $\varepsilon > 0$, the Rellich-Kondrachov compact embedding theorem ensures the existence of at least one positive solution and of infinitely many sign changing solutions. In [11, 18, 20, 26, 28] it was proved that, as $\varepsilon \to 0^+$, the least energy positive solution blows up and concentrates at a point ξ which is a

T. B. has been supported by DAAD project 50766047, Vigoni Project E65E06000080001.

T. D. and A. P. have been supported by the Italian PRIN Research Project 2009 *Metodi variazionali e topologici nello studio dei fenomeni non lineari.*

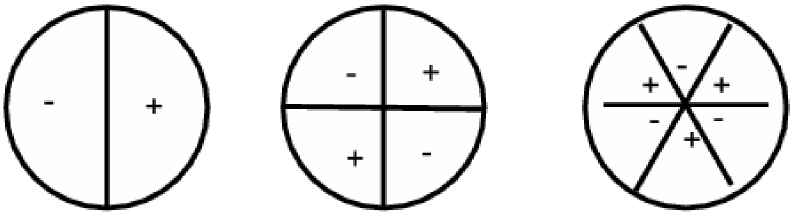

FIGURE 1. The nodal structure of the solutions with 2 peaks, 4 peaks, 6 peaks found in [7].

critical point of the Robin's function of Ω. Successively, in [4, 27] it was studied the existence of positive solutions of (1.1) with $k \geq 2$ blow-up points. In particular, in [17] it was proved that in a convex domain problem (1.1) does not admit any positive solution blowing up at $k \geq 2$ points.

Concerning sign-changing solutions we mention the papers [5, 8, 9] where the authors provide existence and multiplicity of sign-changing solutions for more general problems than (1.1). However, these papers are not concerned with the nodal structure of the solutions. The question whether the nodal surface

$$\mathcal{N}(u) := \operatorname{clos}\{x \in \Omega : u(x) = 0\}$$

of a solution u of (1.1) intersects the boundary $\partial\Omega$ has been investigated in [1]. The number and shape of nodal domains is also important when one investigates competing species or phase separation problems in systems like

$$\begin{cases} -\Delta v = |v|^{2^*-2-\epsilon}v + \beta G_v(v,w), \\ -\Delta w = |w|^{2^*-2-\epsilon}w + \beta G_w(v,w), \\ v, w \in H_0^1(\Omega). \end{cases} \tag{1.2}$$

It has been proved in [13], under appropriate conditions on G modeling the competition, that as $\beta \to -\infty$ positive solutions (v_β, w_β) of (1.2) converge towards (v, w), such that $v \cdot w = 0$ and $u = v - w$ solves (1.1). Thus the nodal domains of u correspond to the domains where the competing species v, w live.

In [7] a solution with exactly one positive and one negative blow-up point is constructed for problem (1.1) if $\varepsilon > 0$ is sufficiently small. The location of the two blow-up points is also characterized and depends on the geometry of the domain. Moreover, the authors proved that when Ω is a ball, for any integer k there exists a solution with k positive peaks and k negative peaks which are located at the vertices of a regular polygon. In particular, the nodal regions of these solutions always intersect the boundary (see Fig. 1).

All the previous results deal with solutions with many simple blow up points.. The presence of sign-changing solutions with a multiple blow-up point is observed in [22] and [24] for problem (1.1). In these papers the authors obtain solutions which have the shape of towers of alternating-sign bubbles, i. e. they are constructed as superpositions of positive bubbles and negative bubbles blowing-up at the same point with a different concentration rate. As a consequence, the nodal regions of these solutions shrink to the blow up point as ε goes to zero. We also mention the paper [10], where the authors study the blow-up of the low energy sign-changing solutions of problem (1.1) and they classify these solutions according to the concentration speeds of the positive and negative part. In particular, they obtain some

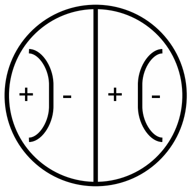

FIGURE 2. The nodal structure of the solution with 4 peaks from [6].

qualitative results, such as symmetry or location of the concentration points when the domain is a ball.

Recently, the authors investigated the existence of solutions in a convex and symmetric domain blowing up positively at k points and negatively at l different points which are aligned with alternating sign along the symmetry axis as $\varepsilon \to 0^+$. The case $k = l = 1$ has been settled in [7], the case $k = l = 2$ in [6]; (see Fig. 2). Other cases, where $k = l \geq 3$ or $k = l - 1$, remained open. In fact, this type of solutions seems to be very hard to find.

In the present paper we are able to prove the existence of a sign changing solution which blows up positively at one point and negatively at two points when $\Omega = \mathcal{B}$ is the open unit ball in $\mathbb{R}^N$. In order to formulate the result we introduce the functions

$$U_{\delta,\xi}(x) = \alpha_N \left(\frac{\delta}{\delta^2 + |x-\xi|^2} \right)^{(N-2)/2}, \quad \alpha_N = (N(N-2))^{(N-2)/4}, \tag{1.3}$$

where $\delta > 0$ and $\xi \in \mathbb{R}^N$. These are actually all positive solutions of the limiting equation

$$-\Delta U = U^{2^*-1} \text{ in } \mathbb{R}^N$$

and constitute the extremals for the Sobolev's critical embedding (see [2, 12, 29]). Let $P : \mathcal{D}^{1,2}(\mathbb{R}^N) \to H_0^1(\mathcal{B})$ denote the orthogonal projection with respect to the scalar product $(u, v) = \langle \nabla u, \nabla v \rangle_{L^2}$.

Theorem 1.1. *There exist $\varepsilon_0 > 0$ and for $0 < \varepsilon < \varepsilon_0$ solutions $\pm u_{1,\varepsilon}, \pm u_{2,\varepsilon} \in H_0^1(\mathcal{B})$ of*

$$-\Delta u = |u|^{2^*-2-\epsilon} u \text{ in } \mathcal{B}, \quad u = 0 \text{ on } \partial\mathcal{B}, \tag{1.4}$$

with the following properties.

a) The solutions are nonradial and even with respect to $x_1, \ldots, x_N$, and their limiting behavior as $\varepsilon \to 0$ is of the form

$$u_{i,\varepsilon} = PU_{\gamma_{i,\varepsilon},0} - PU_{\delta_{i,\varepsilon},\xi_{i,\varepsilon}} - PU_{\delta_{i,\varepsilon},-\xi_{i,\varepsilon}} + O(\varepsilon) \quad \text{in } H_0^1(\mathcal{B})$$

where $\gamma_{i,\varepsilon}, \delta_{i,\varepsilon} > 0$ and $\xi_{i,\varepsilon} \in \mathcal{B} \setminus \{0\}$.

b) $\gamma_{i,\varepsilon}/\varepsilon$ and $\delta_{i,\varepsilon}/\varepsilon$ are bounded away from 0 and ∞, hence the solution $u_{i,\varepsilon}$ has one positive blow-up point at 0 and two negative blow-up points at $\pm\xi_{i,\varepsilon}$.

c) The blow-up points $\xi_{i,\varepsilon}$ are bounded away from 0 and $\partial\mathcal{B}$, and they satisfy $0 < |\xi_{1,\varepsilon}| < |\xi_{2,\varepsilon}| < 1$.

d) The exterior normal derivative of $u_{1,\varepsilon}$ changes sign on $\partial\mathcal{B}$, whereas the exterior normal derivative of $u_{2,\varepsilon}$ is strictly positive on $\partial\mathcal{B}$.

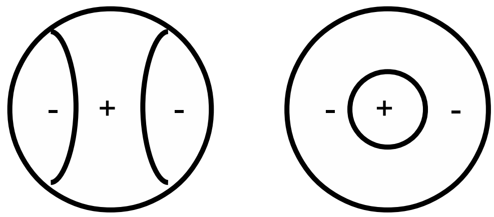

FIGURE 3. The nodal structure of the two solutions with 3 peaks from Theorem 1.1.

Remark 1.2. *a) As a consequence of Theorem 1.1 d) the nodal surface* $\mathcal{N}(u_{1,\varepsilon})$ *of* $u_{1,\varepsilon}$ *intersects the boundary, and the nodal surface* $\mathcal{N}(u_{2,\varepsilon})$ *of* $u_{2,\varepsilon}$ *does not intersect the boundary (see Fig.* (3)*).*

b) Since the problem is radially invariant, for any $A \in O(N)$ *the function* $u \circ A$ *solves* (1.4) *if* u *does. Hence the solutions and the blow-up points* $\pm\xi_{i,\varepsilon}$ *are determined up to rotations.*

c) The proof shows that the solutions $(\varepsilon, u_{1,\varepsilon}), (\varepsilon, u_{2,\varepsilon})$ *lie on connected sets* $\mathcal{C}_1, \mathcal{C}_2 \subset (0, \varepsilon_0) \times H_0^1(\mathcal{B})$*, respectively. We expect that* $u_{1,\varepsilon}, u_{2,\varepsilon}$ *are the only solutions of* (1.4) *having the form as described in Theorem 1.1, and consequently, that the maps*

$$u_i : (0, \varepsilon_0) \to H_0^1(\mathcal{B}), \quad \varepsilon \mapsto u_{i,\varepsilon}, \qquad (i = 1, 2)$$

are continuous; see Remarks 2.4, 3.2.

d) The energy of $u_{1,\varepsilon}$ *is larger than the energy of* $u_{2,\varepsilon}$*. The Morse indices* $m(u_{i,\varepsilon})$ *of the solutions differ by* 1*, i. e.* $m(u_{1,\varepsilon}) = m(u_{2,\varepsilon}) + 1$*. It follows from* [1, Theorem 1.2] *that* $m(u_{2,\varepsilon}) \geq N + 1$*. From a Morse theoretic point of view the topology generated by* $u_{2,\varepsilon}$*, the solution with smaller energy, is canceled by* $u_{1,\varepsilon}$*.*

The proof of Theorem 1.1 relies on a well known Ljapunov-Schmidt procedure. This will be recalled in Section 2 where we reduce the problem to a finite-dimensional one. In Section 3 we prove the existence of two critical points (one local minimum point and one local saddle point) of the reduced energy. These two critical points generate the two solutions of problem (1.4). Finally, in Section 4 we show that the normal derivative of the solution generated by the minimum point changes sign on the boundary, while the normal derivative of the solution generated by the saddle point does not change sign on the boundary of the ball.

It seems very hard to generalize this result to more general domains, except for small perturbations of the ball. Even the case of an an ellipsoid seems to be very difficult. In fact, in the case of the ball the Green's function of the Laplace operator is known and it allows to find the two local critical points of the reduced energy essentially by direct computations. In general, the Green's function is not explicitely known and so it becomes very difficult to find critical points of local type, i. e. local minima or local saddle points. A global min-max scheme to obtain these solutions does not seem to be possible according to Remark 1.2 d).

2. SETTING OF THE PROBLEM

The proof of Theorem 1.1 is based on a *finite dimensional reduction* procedure. We sketch the procedure here and refer to [7] for details. To begin with we introduce scaled versions of the functions from (1.3):

$$U^{\varepsilon}_{\lambda,\rho} = U_{\lambda^2\varepsilon,(\rho,0)}, \quad \lambda > 0,\ \rho \in (0,1);$$

here $(\rho,0) \in \mathcal{B} \subset \mathbb{R} \times \mathbb{R}^{n-1}$. We look for symmetric solutions of (1.4) of the form

$$u_\varepsilon := PU^{\varepsilon}_{\lambda,0} - PU^{\varepsilon}_{\mu,\rho} - PU^{\varepsilon}_{\mu,-\rho} + \phi \tag{2.1}$$

with $\lambda, \mu > 0$, $\rho \in (0,1)$, and $\phi = O(\varepsilon)$. Moreover, ϕ belongs to a suitable space defined below. It is useful to recall that

$$PU_{\delta,\xi}(x) = U_{\delta,\xi}(x) - \gamma_N \delta^{(N-2)/2} H(x,\xi) + O\left(\delta^{(N+2)/2}/(\mathrm{dist}(\xi,\partial\mathcal{B}))^N\right)$$

where $\gamma_N > 0$ is a constant and the function H is the regular part of the the Green's function G of the Laplace operator in the ball, i. e.

$$G(x,y) = \frac{1}{|x-y|^{N-2}} - H(x,y), \qquad H(x,y) = \frac{1}{(|x|^2|y|^2 + 1 - 2(x,y))^{(N-2)/2}}\ .$$

By the principle of symmetric criticality critical points of the energy functional

$$J_\varepsilon : H^1_0(\mathcal{B}) \to \mathbb{R}, \quad J_\varepsilon(u) = \frac{1}{2}\int_{\mathcal{B}} |\nabla u|^2 dx - \frac{1}{2^* - \varepsilon}\int_{\mathcal{B}} |u|^{2^*-\varepsilon} dx,$$

constrained to the subspace

$$H_e := \{u \in H^1_0(\mathcal{B}) : u \text{ is even in } x_1, \dots, x_N\} \subset H^1_0(\mathcal{B}),$$

are solutions to problem (1.4). In order to define the space for ϕ we set

$$K^{\varepsilon}_{\lambda,\mu,\rho} := \mathrm{span}\left\{ P\left(\frac{\partial}{\partial\lambda}U^{\varepsilon}_{\lambda,0}\right),\ P\left(\frac{\partial}{\partial\mu}U^{\varepsilon}_{\mu,\rho}\right),\ P\left(\frac{\partial}{\partial\rho}U^{\varepsilon}_{\lambda,\rho}\right)\right\} \subset H_e,$$

and

$$\left(K^{\varepsilon}_{\lambda,\mu,\rho}\right)^{\perp} := \left\{\phi \in H_e : (\phi,\psi) = 0 \text{ for any } \psi \in K^{\varepsilon}_{\lambda,\mu,\rho}\right\} \subset H_e.$$

Here $(u,v) = \langle \nabla u, \nabla v\rangle_{L^2}$ will be used as inner product on the Hilbert space H_e. We write $\|u\| = \|\nabla u\|_{L^2}$ for the associated norm on H_e.

We first solve an intermediate problem for ϕ (see [7]). Define

$$V^{\varepsilon}_{\lambda,\mu,\rho} := PU^{\varepsilon}_{\lambda,0} - PU^{\varepsilon}_{\mu,\rho} - PU^{\varepsilon}_{\mu,-\rho},$$

with $(\lambda,\mu,\rho) \in \mathcal{D} := (0,\infty) \times (0,\infty) \times (0,1)$, and, for any $\eta > 0$ small,

$$\mathcal{D}_\eta := (\eta, \eta^{-1}) \times (\eta,\eta^{-1}) \times (\eta, 1-\eta) \subset D.$$

Lemma 2.1. *For $\eta > 0$ small there exists $\varepsilon_1 > 0$ and a constant $C > 0$ such that for each $\varepsilon \in (0,\varepsilon_1)$ and each $(\lambda,\mu,\rho) \in \mathcal{D}_\eta$ there exists a unique $\phi = \phi^{\varepsilon}_{\lambda,\mu,\rho} \in \left(K^{\varepsilon}_{\lambda,\mu,\rho}\right)^{\perp}$ satisfying*

$$\Delta(V^{\varepsilon}_{\lambda,\mu,\rho} + \phi) + |V^{\varepsilon}_{\lambda,\mu,\rho} + \phi|^{2^*-2-\varepsilon}(V^{\varepsilon}_{\lambda,\mu,\rho} + \phi) \in K^{\varepsilon}_{\lambda,\mu,\rho}$$

and $\|\phi\| < C\varepsilon$. Moreover, the map $D_\eta \ni (\lambda,\mu,\rho) \mapsto \phi^{\varepsilon}_{\lambda,\mu,\rho} \in H_e$ is of class C^1.

Now we introduce the reduced energy functional

$$\widetilde{J}_\varepsilon : \mathcal{D}_\eta \to \mathbb{R}, \quad \widetilde{J}_\varepsilon(\lambda,\mu,\rho) := J_\varepsilon\left(V^{\varepsilon}_{\lambda,\mu,\rho} + \phi^{\varepsilon}_{\lambda,\mu,\rho}\right),$$

where $\phi^{\varepsilon}_{\lambda,\mu,\rho}$ has been constructed in Lemma 2.1. The next result (see [4]) reduces the original problem (1.4) to a finite dimensional one.

Proposition 2.2. *The point $(\lambda,\mu,\rho) \in \mathcal{D}_\eta$ is a critical point of $\widetilde{J}_\varepsilon$ if, and only if, the corresponding function $u_\varepsilon = V^{\varepsilon}_{\lambda,\mu,\rho} + \phi^{\varepsilon}_{\lambda,\mu,\rho}$ is a solution of (1.4).*

Now we expand the reduced energy.

Proposition 2.3. *We have*

$$\widetilde{J}_\varepsilon(\lambda,\mu,\rho) = c_{1N} + c_{2N}\varepsilon\log\varepsilon + c_{3N}\varepsilon + c_{4N}\varepsilon F(\lambda,\mu,\rho) + o(\varepsilon) \tag{2.2}$$

$\mathcal{C}^1$-uniformly on compact sets of $\mathcal{D}$. Here the c_{iN}'s are positive constants which depend only on N. Moreover

$$F(\lambda,\mu,\rho) := a\lambda^2 + 2\mu^2\alpha(\rho) + 4\lambda\mu\beta(\rho) - c_n\ln\lambda - 2c_n\ln\mu, \tag{2.3}$$

with c_N a positive constant depending only on N, and $a := H(0,0) = 1$,

$$\begin{aligned}\alpha(\rho) &:= H((\rho,0),(\rho,0)) - G((\rho,0),(-\rho,0))\\ &= \frac{1}{(1-\rho^2)^{n-2}} - \frac{1}{(2\rho)^{N-2}} + \frac{1}{(1+\rho^2)^{N-2}},\\ \beta(\rho) &:= G((\rho,0),(0,0)) = \frac{1}{\rho^{N-2}} - 1.\end{aligned}$$

Proof. The proof proceeds as in [15, 16]; see also [7, Proposition 3.1]. □

Remark 2.4. *If $N < 6$ then the original functional is of class $\mathcal{C}^3$ for ε small, hence the map $D_\eta \ni (\lambda,\mu,\rho) \mapsto \phi^\varepsilon_{\lambda,\mu,\rho} \in H_e$ from Lemma 2.1 is of class $\mathcal{C}^2$. Then the reduced functional $\widetilde{J}_\varepsilon$ is of class $\mathcal{C}^2$, and it can be proved as in* [19, Proposition 2.3] *that the expansion in Proposition 2.3 is $\mathcal{C}^2$-uniformly on compact sets of $\mathcal{D}$.*

Our main result will follow from the following proposition applied to $\nabla\widetilde{J}_\varepsilon$.

Proposition 2.5. *Let $U \subset \mathbb{R}^m$ be open and bounded, and consider a one parameter family of maps $h_\varepsilon : \overline{U} \to \mathbb{R}^m$ of the form $h_\varepsilon(x) = \varepsilon f(x) + g_\varepsilon(x)$ with $f, g_\varepsilon : \overline{U} \to \mathbb{R}^m$ of class $\mathcal{C}^k$, $k \ge 0$. Suppose the map $(0,\varepsilon_1) \to \mathcal{C}^k(\overline{U},\mathbb{R}^m)$, $\varepsilon \mapsto g_\varepsilon$, is $\mathcal{C}^k$ and satisfies $\|g_\varepsilon\|_{\mathcal{C}^k} = o(\varepsilon)$ as $\varepsilon \to 0$.*

a) *If the Brouwer degree* $\deg(f,U,0) \ne 0$ *is well defined and nontrivial then there exists $\varepsilon_0 > 0$ and a connected subset $\mathcal{C} \subset (0,\varepsilon_0) \times U$ with the following properties:*

(i) *$\mathcal{C}$ covers the interval $(0,\varepsilon_0)$, i. e. for every $0 < \varepsilon < \varepsilon_0$ there exists $x_\varepsilon \in U$ with $(\varepsilon, x_\varepsilon) \in \mathcal{C}$.*
(ii) *If $(\varepsilon, x) \in \mathcal{C}$ then $h_\varepsilon(x) = 0$.*
(iii) *Given a sequence $(\varepsilon_n, x_n) \in \mathcal{C}$ with $\varepsilon_n \to 0$, $x_n \to x_0$, then $f(x_0) = 0$.*

b) *If $k = 1$ and if $x_0 \in U$ is a nondegenerate zero of f then there exists $\varepsilon_0 > 0$ and a $\mathcal{C}^1$-map $(0,\varepsilon_0) \to U$, $\varepsilon \mapsto x_\varepsilon$, such that $h_\varepsilon(x_\varepsilon) = 0$ for all $\varepsilon \in (0,\varepsilon_0)$. Moreover, $x_\varepsilon \to x_0$ as $\varepsilon \to 0$.*

Proof. a) The maps $\varepsilon^{-1}h_\varepsilon = f + \varepsilon^{-1}g_\varepsilon$ converge uniformly towards f as $\varepsilon \to 0$, hence $\deg(\varepsilon^{-1}h_\varepsilon, U, 0) = \deg(f,U,0) \ne 0$ is well defined and nontrivial for $\varepsilon > 0$ small. The existence and the properties of $\mathcal{C}$ follow by standard arguments.

b) Existence and uniqueness of x_ε follow from the contraction mapping principle applied to

$$f_\varepsilon := \mathrm{id} - \varepsilon^{-1}Df(x_0)^{-1} \circ h_\varepsilon = \mathrm{id} - Df(x_0)^{-1} \circ f - \varepsilon^{-1}Df(x_0)^{-1} \circ g_\varepsilon.$$

For $\delta, \varepsilon > 0$ small $f_\varepsilon : B_\delta(x_0) \to B_\delta(x_0)$ is well defined and a contraction. Differentiability is a consequence of the implicit function theorem. □

Finally, the propositions 2.3, 2.5, and [7, Lemma 4.5] imply the following existence theorem. Observe that F is analytic, hence a critical point (λ,μ,ρ) of F is automatically isolated, and its degree, $\deg(F,(\lambda,\mu,\rho)) := \deg(\nabla F, U_\delta(\lambda,\mu,\rho), 0) \in \mathbb{Z}$ for $\delta > 0$ small, is well defined.

Theorem 2.6. *Let $(\tilde\lambda, \tilde\mu, \tilde\rho)$ be a critical point of F with nontrivial degree.*

a) There exist $\tilde\varepsilon > 0$ and a connected subset $\widetilde{\mathcal{C}} \subset (0,\tilde\varepsilon) \times \mathcal{D}$ with the following properties:

(i) *$\widetilde{\mathcal{C}}$ covers the interval $(0, \tilde{\varepsilon})$.*
(ii) *If $(\varepsilon, \lambda, \mu, \rho) \in \widetilde{\mathcal{C}}$ then $\nabla \widetilde{J}_\varepsilon(\lambda, \mu, \rho) = 0$.*
(iii) *Given a sequence $(\varepsilon_n, \lambda_n, \mu_n, \rho_n) \in \widetilde{\mathcal{C}}$ with $\varepsilon_n \to 0$, then $(\lambda_n, \mu_n, \rho_n) \to (\tilde{\lambda}, \tilde{\mu}, \tilde{\rho})$.*
(iv) *Setting*

$$\mathcal{C} := \{(\varepsilon, u) : u = V^\varepsilon_{\lambda,\mu,\rho} + \phi^\varepsilon_{\lambda,\mu,\rho},\ (\varepsilon, \lambda, \mu, \rho) \in \widetilde{\mathcal{C}}\} \subset (0, \tilde{\varepsilon}) \times H^1_0(\mathcal{B})$$

then any family $(\varepsilon, u_\varepsilon) \in \mathcal{C}$ converges in $C^1_{loc}\left(\overline{\Omega} \setminus \{(0,0), (\tilde{\rho}, 0), (-\tilde{\rho}, 0)\}\right)$

$$\frac{1}{\sqrt{\varepsilon}} u_\varepsilon(x) \to \alpha_N \left(\tilde{\lambda} G(x,0) - \tilde{\mu} G\left(x, (\tilde{\rho}, 0)\right) - \tilde{\mu} G\left(x, (-\tilde{\rho}, 0)\right)\right) \quad \text{as } \varepsilon \to 0. \tag{2.4}$$

b) If $(\tilde{\lambda}, \tilde{\mu}, \tilde{\rho})$ is a nondegenerate critical point of F then the set $\widetilde{\mathcal{C}}$ from a) is the graph of a $\mathcal{C}^1$-map $(0, \tilde{\varepsilon}) \to \mathcal{D}$. Correspondingly, the set $\mathcal{C}$ is the graph of a $\mathcal{C}^1$-map $(0, \tilde{\varepsilon}) \to H^1_0(\mathcal{B})$.

3. The existence of two solutions

In this section we will prove Theorem 1.1 a), b), c).

Lemma 3.1. *The function F from (2.3) has two isolated critical points $(\lambda_1, \mu_1, \rho_1)$ and $(\lambda_2, \mu_2, \rho_2)$. The first one is a local saddle point with Morse index 1, hence it has degree $\deg(F, (\lambda_1, \mu_1, \rho_1)) = -1$. The second is a strict local minimum, hence $\deg(F, (\lambda_1, \mu_1, \rho_1)) = 1$. Moreover, $\rho_1 < \frac{1}{2} < \rho_2$.*

Postponing the proof of this lemma we first deduce the

Proof of Theorem 1.1 a), b), c). Let $u_{i,\varepsilon}$ be the solution of (1.4) corresponding to the critical point $(\lambda_i, \mu_i, \rho_i)$ of F from Lemma 3.1. Then the blow-up property of Theorem 1.1 a) is satisfied by construction (see Theorem 2.6) with

$$\xi_{i,\varepsilon} \to (\rho_i, 0), \ \frac{\gamma_{i,\varepsilon}}{\varepsilon} \to \sqrt{\lambda_i}, \ \frac{\delta_{i,\varepsilon}}{\varepsilon} \to \sqrt{\mu_i},$$

as $\varepsilon \to 0$. Properties b) and c) of Theorem 1.1 follow immediately. □

Theorem 1.1 d) will be proved in the next section.

Remark 3.2. *We conjecture that F has precisely two critical points. Then we obtain continuous curves*

$$(0, \varepsilon_0) \to H_e \subset H^1_0(\mathcal{B}), \quad \varepsilon \mapsto u_{i,\varepsilon}, \qquad (i = 1, 2)$$

of sign changing solutions $u_{i,\varepsilon}$ to problem (1.4) as stated in Remark 1.2. If $N < 6$ we checked numerically that the two critical points from Lemma 3.1 are nondegenerate, hence Theorem 2.6 implies that the two curves are of class $\mathcal{C}^1$.

Proof of Lemma 3.1. First of all, it is useful to point out that, since $\alpha' > 0$, $\alpha(\rho) \to -\infty$ as $\rho \to 0$, $\alpha(\frac{1}{2}) > 0$, there exists $\rho_0 \in (0, \frac{1}{2})$ such that

$$\alpha(\rho_0) = 0 \text{ and } \alpha(\rho) > 0 \text{ for any } \rho \in (\rho_0, 1).$$

We have

$$\partial_\lambda F(\lambda, \mu, \rho) = 2\lambda + 4\mu\beta(\rho) - \frac{c_N}{\lambda}, \quad \partial_\mu F(\lambda, \mu, \rho) = 4\mu\alpha(\rho) + 4\lambda\beta(\rho) - \frac{2c_N}{\mu}.$$

Then for any $\rho \in (\rho_0, 1)$ there exist unique $\lambda(\rho)$ and $\mu(\rho)$ such that

$$\nabla_{\lambda,\mu} F(\lambda(\rho), \mu(\rho), \rho) = 0.$$

More precisely,

$$\mu(\rho)=\sqrt{\frac{c_N}{2\alpha(\rho)+2\Lambda(\rho)\beta(\rho)}},\qquad \lambda(\rho)=\Lambda(\rho)\mu(\rho) \tag{3.1}$$

where

$$\Lambda(\rho):=\frac{\sqrt{\beta^2(\rho)+4\alpha(\rho)}-\beta(\rho)}{2}>0. \tag{3.2}$$

We remark that the Hesse matrix $D^2_{\lambda,\mu}F(\lambda(\rho),\mu(\rho),\rho)$ is positively definite and in particular non-degenerate. In fact, since $\nabla_{\lambda,\mu}F(\lambda(\rho),\mu(\rho),\rho)=0$, an easy computation shows that

$$\begin{aligned} D^2_{\lambda,\mu}F(\lambda(\rho),\mu(\rho),\rho) &= \begin{pmatrix} 2+\dfrac{c_N}{(\lambda(\rho))^2} & 4\beta(\rho) \\ 4\beta(\rho) & 4\alpha(\rho)+\dfrac{2c_N}{(\mu(\rho))^2} \end{pmatrix} \\ &= 16\begin{pmatrix} 1+\dfrac{\beta(\rho)}{\Lambda(\rho)} & \beta(\rho) \\ \beta(\rho) & 2\alpha(\rho)+\Lambda(\rho)\beta(\rho) \end{pmatrix}. \end{aligned}$$

Then $\mathrm{tr}D^2_{\lambda,\mu}F(\lambda(\rho),\mu(\rho),\rho)>0$ and

$$\det D^2_{\lambda,\mu}F(\lambda(\rho),\mu(\rho),\rho)=16\left(2\alpha(\rho)+\Lambda(\rho)\beta(\rho)+\frac{2\alpha(\rho)\beta(\rho)}{\Lambda(\rho)}\right)>0.$$

Now, let us consider the reduced function

$$f(\rho):=F(\lambda(\rho),\mu(\rho),\rho)=\frac{3}{2}c_N-c_N\log\left[\lambda(\rho)\mu^2(\rho)\right],\quad \rho\in(\rho_0,1). \tag{3.3}$$

Since $f:(\rho_0,1)\to\mathbb{R}$ is analytic, critical points are either strict local maxima or minima. If ρ_1 is a local maximum of f then $(\lambda(\rho_1),\mu(\rho_1),\rho_1)$ is a critical point of F with Morse index 1 and degree -1. If ρ_2 is a local minimum of f then $(\lambda(\rho_2),\mu(\rho_2),\rho_2)$ is a local minimum of F with degree $+1$.

Observe that

$$\lim_{\rho\to\rho_0^+}f(\rho)=-\infty\quad\text{and}\quad\lim_{\rho\to1^-}f(\rho)=+\infty. \tag{3.4}$$

In fact, since $\alpha(\rho_0)=0$ we get as $\rho\to\rho_0^+$

$$\Lambda(\rho)\sim\alpha(\rho),\quad \mu(\rho)\sim(\alpha(\rho))^{-1/2},\quad \lambda(\rho)\sim(\alpha(\rho))^{1/2}$$

which imply

$$\lambda(\rho)\mu^2(\rho)\to+\infty\quad\text{and}\quad f(\rho)\to-\infty\quad\text{as }\rho\to\rho_0^+.$$

Moreover, as $\rho\to1^-$ we have

$$\alpha(\rho)\sim\frac{1}{(1-\rho)^{N-2}},\quad \beta(\rho)\sim-\rho,\quad \Lambda(\rho)\sim\sqrt{\alpha(\rho)}\sim\frac{1}{(1-\rho)^{\frac{N-2}{2}}},$$

which imply

$$\mu(\rho)\sim(1-\rho)^{\frac{N-2}{2}},\quad \lambda(\rho)\sim1\quad\text{and}\quad f(\rho)\to+\infty\quad\text{as }\rho\to1^-.$$

We claim that

$$f'\left(\frac12\right)<0\qquad\text{for all }N\geq3. \tag{3.5}$$

Then it follows that f has a local maximum point $\rho_1<\frac12$ and a local minimum point $\rho_2>\frac12$.

An easy computation shows that

$$\begin{aligned}f'(\rho) &= \partial_\lambda F(\lambda(\rho),\mu(\rho),\rho)\lambda'(\rho) + \partial_\mu F(\lambda(\rho),\mu(\rho),\rho)\mu'(\rho) + \partial_\rho F(\lambda(\rho),\mu(\rho),\rho)\\ &= \partial_\rho F(\lambda(\rho),\mu(\rho),\rho)\\ &= 2\left(\mu(\rho)\right)^2\alpha'(\rho) + 4\mu(\rho)\lambda(\rho)\beta'(\rho) = 2\left(\mu(\rho)\right)^2\left(\alpha'(\rho)+2\Lambda(\rho)\beta'(\rho)\right).\end{aligned}$$

Thus setting

$$\chi(\rho) := \frac{f'(\rho)}{2\left(\mu(\rho)\right)^2} = \alpha'(\rho) + 2\Lambda(\rho)\beta'(\rho) \tag{3.6}$$

we need to show that $\chi(\frac12) < 0$.

For $N = 3$ this can be checked explicitely. Next observe that

$$\alpha\Big(\frac12\Big) < \beta^2\Big(\frac12\Big) \quad \text{if } N \geq 4 \tag{3.7}$$

because

$$\begin{aligned}\alpha\Big(\frac12\Big) &= \Big(\frac43\Big)^{N-2} - 1 + \Big(\frac45\Big)^{N-2} \leq \Big(\frac43\Big)^{N-2} \leq \frac14 2^{N-1}\\ &\leq \frac14\left(2^{N-3}(2^{N-1}-4)+1\right) = \frac14\beta^2\Big(\frac12\Big) \quad \text{if } N \geq 4.\end{aligned}$$

Then, by (3.7), using the inequality

$$\sqrt{1+x} - 1 \geq \frac25 x \quad \text{for any } x \in (0,1), \tag{3.8}$$

for $N \geq 4$ we get

$$\begin{aligned}\chi\Big(\frac12\Big) &= \alpha'\Big(\frac12\Big) + \beta\Big(\frac12\Big)\Bigg(\sqrt{1+\frac{4\alpha(\frac12)}{\beta^2(\frac12)}} - 1\Bigg)\beta'\Big(\frac12\Big)\\ &\leq \alpha'\Big(\frac12\Big) + \frac85\frac{\alpha(\frac12)}{\beta(\frac12)}\beta'\Big(\frac12\Big).\end{aligned} \tag{3.9}$$

We compute

$$\frac{1}{N-2}\alpha'\Big(\frac12\Big) = \Big(\frac43\Big)^{N-1} + 2 - \Big(\frac45\Big)^{N-1}$$

and

$$\frac{1}{N-2}\beta'\Big(\frac12\Big) = -2^{N-1},$$

and

$$\frac{\alpha(\frac12)}{\beta(\frac12)} = \frac{(\frac43)^{N-2} - 1 + (\frac45)^{N-2}}{2^{N-2}-1} \geq \frac{(\frac43)^{N-2} - 1 + (\frac45)^{N-2}}{2^{N-2}}.$$

Therefore, by (3.9), for $N \geq 4$ we obtain

$$\begin{aligned}\frac{1}{N-2}\chi\Big(\frac12\Big) &\leq \frac{1}{N-2}\alpha'\Big(\frac12\Big) + \frac{1}{N-2}\frac85\frac{\alpha(\frac12)}{\beta(\frac12)}\beta'(\frac12)\\ &\leq \Big(\frac43\Big)^{N-1} + 2 - \Big(\frac45\Big)^{N-1} - \frac85\frac{(\frac43)^{N-2} - 1 + (\frac45)^{N-2}}{2^{N-2}}2^{n-1}\\ &= -\frac{28}{15}\Big(\frac43\Big)^{N-2} - 4\Big(\frac45\Big)^{N-2} + \frac{26}{5}.\end{aligned}$$

We observe that $\frac{28}{15}(\frac43)^{N-2} > \frac{26}{5}$ for $N \geq 6$, while, by a direct computation, $-\frac{28}{15}(\frac43)^{N-2} - 4(\frac45)^{N-2} + \frac{26}{5} < 0$ for $N = 4,5$. This allows us to conclude that $\chi(\frac12) < 0$ if $N \geq 4$, hence (3.5) holds and Lemma 3.1 follows. □

For the profile of the second solution we also need the estimate

$$f'(\bar\rho) < 0 \quad \text{for } \bar\rho := \frac{\sqrt5 - 1}{2}, \text{ for all } N \geq 3. \tag{3.10}$$

It is sufficient to prove the inequality $\chi(\bar\rho) < 0$, which can be checked for $N = 3, 4, 5$ by a direct computation. For the case $N \geq 6$ we first observe that

$$\sqrt{1+t} - 1 \geq \frac{t}{3} \quad \forall\, t \in [0,3], \tag{3.11}$$

and

$$1 + \bar\rho^2 \geq 2\bar\rho \geq \bar\rho\sqrt{1+\bar\rho^2} \geq 1 - \bar\rho^2 = \bar\rho. \tag{3.12}$$

In order to use (3.11) we need to check that

$$\frac{4\alpha(\bar\rho)}{\beta^2(\bar\rho)} \leq 3 \quad \text{if } N \geq 6. \tag{3.13}$$

In fact, (3.12) implies

$$\begin{aligned}
&4\alpha(\bar\rho) - 3\beta^2(\bar\rho)\\
&\quad = 4\left(\frac{1}{(1-\bar\rho^2)^{N-2}} - \frac{1}{(2\bar\rho)^{N-2}} + \frac{1}{(1+\bar\rho^2)^{N-2}}\right) - 3\left(\frac{1}{\bar\rho^{2N-4}} + 1 - 2\frac{1}{\bar\rho^{N-2}}\right)\\
&\quad \leq \frac{10}{\bar\rho^{N-2}} - \frac{3}{\bar\rho^{2N-4}}
\end{aligned}$$

and so (3.13) follows because $\bar\rho < \left(\frac{3}{10}\right)^{N-2}$ for $N \geq 6$.

Therefore, by the definition of Λ in (3.2), using (3.11) and (3.13) we deduce

$$\Lambda(\bar\rho) \geq \frac{2\alpha(\bar\rho)}{3\beta(\bar\rho)} > \frac{2}{3}\alpha(\bar\rho)\bar\rho^{N-2} \quad \text{for } N \geq 6. \tag{3.14}$$

Now, (3.12) and (3.14) combined with the definition of χ in (3.6) imply for $N \geq 6$:

$$\begin{aligned}
\chi(\bar\rho) &\leq \alpha'(\bar\rho) - \frac{4}{3}(N-2)\frac{\alpha(\bar\rho)}{\bar\rho}\\
&= \frac{2(N-2)}{3\bar\rho}\left(\frac{5\bar\rho^2 - 2}{(1-\bar\rho^2)^{N-1}} + \frac{7\bar\rho}{(2\bar\rho)^{N-1}} - \frac{5\bar\rho^2 + 2}{(1+\bar\rho^2)^{N-1}}\right)\\
&= \frac{2(N-2)}{3\bar\rho^n}\left(5\bar\rho^2 - 2 + \frac{7\bar\rho}{2^{N-1}} - (5\bar\rho^2+2)\left(\frac{\bar\rho}{1+\bar\rho^2}\right)^{N-1}\right) < 0.
\end{aligned}$$

The last inequality follows for $N = 6$ by an explicit computation. For $N \geq 7$ observe that the term $5\bar\rho^2 - 2 + \frac{7\bar\rho}{2^{n-1}}$ decreases as N increases, and $5\bar\rho^2 - 2 + \frac{7\bar\rho}{2^{n-1}} < 0$ for $N = 7$.

This finishes the proof of $\chi(\bar\rho) < 0$ for $N \geq 6$, hence (3.10) holds. As a consequence we obtain $\rho_2 > \bar\rho$.

4. The profile of the solutions

In this section we prove Theorem 1.1 d). Let $u_{1\varepsilon}$ and $u_{2\varepsilon}$ be the solutions generated by the critical points $(\lambda(\rho_1), \mu(\rho_1), \rho_1)$ and $(\lambda(\rho_2), \mu(\rho_2), \rho_2)$ of F, respectively, as stated in Theorem 2.6. Here $\lambda(\rho)$ and $\mu(\rho)$ are from (3.1), and ρ_1, ρ_2 are the two critical points of the reduced function f from (3.3). Recall that $0 < \rho_0 < \rho_1 < \frac{1}{2}$ and $\bar\rho = \frac{\sqrt5 - 1}{2} < \rho_2 < 1$.

As a consequence of (2.4) we deduce that in a neighborhood $\mathfrak{U}$ of $\partial\mathcal{B}$ not containing the blow-up points there holds

$$\frac{u_{i\epsilon}(x)}{\alpha_N\mu_i\sqrt{\varepsilon}} \to \varphi(\rho_i, x) := \Lambda(\rho_i)G(x,0) - \big(G(x,(\rho_i,0)) + G(x,(-\rho_i,0))\big) \ \text{ in } C^1(\mathfrak{U})$$

as ε goes to zero. Here Λ has been defined in (3.1) and (3.2).

It follows that the exterior normal derivative satisfies

$$\partial_\nu\varphi(\rho_i, x) = (N-2)\psi(\rho_i, x_1), \qquad x_1 \in [-1,1],$$

where

$$\psi(\rho,x_1) := -\Lambda(\rho) + (1-\rho^2)\left(\frac{1}{(\rho^2+1-2\rho x_1)^{N/2}} + \frac{1}{(\rho^2+1+2\rho x_1)^{N/2}}\right). \tag{4.1}$$

Defining

$$M(\rho) := \max_{|x_1|\le 1} \psi(\rho,x_1) = \psi(\rho,1) = -\Lambda(\rho) + (1-\rho^2)\left(\frac{1}{(1-\rho)^N} + \frac{1}{(1+\rho)^N}\right),$$

and

$$m(\rho) := \min_{|x_1|\le 1} \psi(\rho,x_1) = \psi(\rho,0) = -\Lambda(\rho) + 2(1-\rho^2)\frac{1}{(\rho^2+1)^{N/2}}$$

we immediately obtain:

$$m(\rho_i) > 0 \quad \Longrightarrow \quad \partial_\nu u_{i,\varepsilon} \text{ does not change sign in } \partial\Omega,$$

and

$$m(\rho_i) < 0 < M(\rho_i) \quad \Longrightarrow \quad \partial_\nu u_{i,\varepsilon} \text{ does change sign in } \partial\Omega.$$

Thus Theorem 1.1 d) follows if we can show that

$$m(\rho_1) > 0, \tag{4.2}$$

and

$$m(\rho_2) < 0 < M(\rho_2). \tag{4.3}$$

For the proof of these inequalities we first observe that

$$\Lambda' = \frac{\beta'\left(\beta - \sqrt{\beta^2+4\alpha}\right) + 2\alpha'}{2\sqrt{\beta^2+4\alpha}} > 0 \quad \text{for all } \rho \in (\rho_0,1), \tag{4.4}$$

since $\beta' < 0$, $\alpha' > 0$ and $\alpha > 0$ in $(\rho_0,1)$. Moreover, using (4.4), a simple calculation shows that

$$m'(\rho) < 0 \quad \text{for all } \rho \in (\rho_0,1). \tag{4.5}$$

Proof of (4.2). By (4.5) it suffices to prove that

$$m\left(\frac{1}{2}\right) > 0. \tag{4.6}$$

This can be checked for $N=3$ by explicit computation. For $N \ge 4$ we argue as follows. Using the inequality $\sqrt{1+x}-1 < \frac{x}{2}$ for $x>0$ we obtain

$$m\left(\frac{1}{2}\right) > -\frac{\alpha\left(\frac{1}{2}\right)}{\beta(\frac{1}{2})} + \frac{3}{2}\left(\frac{4}{5}\right)^{N/2} = -\frac{\left(\frac{4}{3}\right)^{N-2} - 1 + \left(\frac{4}{5}\right)^{N-2}}{2^{N-2}-1} + \frac{3}{2}\left(\frac{4}{5}\right)^{N/2}$$

Therefore we only need to show that

$$\frac{3}{8}\cdot\left(\frac{4}{\sqrt{5}}\right)^N > \frac{3}{2}\left(\frac{2}{\sqrt{5}}\right)^N + \left(\frac{4}{3}\right)^{N-2} - 1 + \left(\frac{4}{5}\right)^{N-2}.$$

This is easily checked for $N=4$, and then it holds for all $N \ge 4$. □

Proof of (4.3). First of all, we remark that $M(\rho) > 0$ for any ρ such that $\chi(\rho) = 0$ where χ has been defined in (3.6). In fact, $\chi(\rho)=0$ implies $\Lambda(\rho) = -\frac{\alpha'(\rho)}{2\beta'(\rho)}$ and so

$$M(\rho) = \frac{\rho^N}{(1+\rho^2)^{N-1}} + \frac{2^{N-1}\left((1+\rho)^N + (1-\rho)^N - \rho^N\right) - (1-\rho^2)^{N-1}}{2^{N-1}(1-\rho^2)^{N-1}} > 0, \tag{4.7}$$

because a direct calculation shows that

$$2^{N-1}\left((1+\rho)^N + (1-\rho)^N - \rho^N\right) - (1-\rho^2)^{N-1} > 0 \quad \text{for any } \rho \in [0,1].$$

As a consequence we obtain $M(\rho_i) > 0$ because $\chi(\rho_i) = 0$. Now it remains to show that $m(\rho_2) < 0$. Since m is decreasing in ρ and $\rho_2 > \bar\rho$ it suffices to prove $m(\bar\rho) < 0$. Using (3.12) and (3.14) we obtain

$$m(\bar\rho) < -\frac{2}{3}\left(\frac{\bar\rho}{1-\bar\rho^2}\right)^{N-2} + \frac{2}{3}\frac{1}{2^{N-2}} - \frac{2}{3}\left(\frac{\bar\rho}{1+\bar\rho^2}\right)^{N-2} + 2\frac{1-\bar\rho^2}{\left(\sqrt{1+\bar\rho^2}\right)^N}$$
$$\le \frac{2}{3}\left(\frac{\bar\rho}{1-\bar\rho^2}\right)^N \left[-\frac{(1-\bar\rho^2)^2}{\bar\rho^2} + 4\left(\frac{1-\bar\rho^2}{2\bar\rho}\right)^N + 3(1-\bar\rho^2)\left(\frac{1-\bar\rho^2}{\bar\rho\sqrt{1+\bar\rho^2}}\right)^N\right].$$

The last expression is negative for $N = 6$, hence for all $N \ge 6$. For $N = 3, 4, 5$ we show $m(\bar\rho) < 0$ by an explicit computation. □

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