Morse Theory and Nonlinear Differential Equations

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1 Introduction

Classical Morse theory's object is the relation between the topological type of critical points of a function f and the topological structure of the manifold X on which f is defined. Traditionally Morse theory deals with the case where all critical points are non-degenerate and it relates the index of the Hessian of a critical point to the homology of the manifold. A closely related more homotopy theoretical approach goes back to Lusternik and Schnirelmann allowing for degenerate critical points. Nowadays these topics are subsumed under the more general heading of critical point theory. The classical Morse theory in finite dimension is described in [42], first extensions to Hilbert manifolds in [43, 47]. In this survey we treat Morse theory on Hilbert manifolds for functions with degenerate critical points.

In section 2 the general case of a topological space X and a continuous function $f : X \to \mathbb{R}$ is considered.

The topological type of the "critical set" between a and b is described by the Morse polynomial $M_f(t; a, b)$. The relative homology of the pair (f^a, f^b) is described by the Poincaré polynomial $P_f(t; a, b)$. By definition $f^a = \{x \in X : f(x) \leq a\}$. Then there exists a polynomial $Q(t)$ with non-negative integer coefficients such that

$$M_f(t; a, b) = P_f(t; a, b) + (1 + t)Q(t).$$

The Morse inequalities follow from this relation.

In section 3 the Morse polynomial is computed in the case of isolated critical points of finite Morse index. Besides the Morse lemma, the main results are the Shifting Theorem and the Splitting Theorem, due to Gromoll and Meyer. As in [17] and [39] we consider the case of an infinite-dimensional Hilbert space.

In section 4 we give some elementary, but typical, applications to semi-linear elliptic problems. We consider in particular the Dirichlet problem

$$\begin{cases} -\Delta u = g(u), & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega. \end{cases}$$

This section contains also a general bifurcation theorem.

Another typical problem to which Morse theory can be applied is the study of existence of periodic solutions for the second order Newtonian systems of ordinary differential equations

$$-\ddot{q} = V_q(q,t), \quad q \in \mathbb{R}^N.$$

However, since we consider the more general and more difficult case of first order Hamiltonian systems in section 6 in some detail, we do not discuss second order systems here and refer the reader e.g. to [39]. We do however discuss the problem of the existence of closed geodesics on a riemannian manifold. Work on this problem was vital for the development of Morse theory, it still offers open problems, and it contains difficulties also present in the more general problem of first order Hamiltonian systems.

Section 5 contains a Morse theory for functionals of the form

$$\Phi(x) = \frac{1}{2}||x^+||^2 - \frac{1}{2}||x^-||^2 - \psi(x)$$

defined on a Hilbert space E, where $x = x^+ + x^0 + x^- \in E = E^+ \oplus E^0 \oplus E^-$, $\dim E^0 < \infty$, $\dim E^\pm = \infty$ and $\nabla\psi$ is a compact operator. The Morse indices are infinite but using a suitable cohomology theory, the Poincaré and the Morse polynomials can be defined. Theorem 5.7 contains the Morse inequalities. The local theory is also extended to this setting.

Section 6 contains applications of the results of section 5 to Hamiltonian systems. We consider the existence of periodic solutions of the system

$$\dot{z} = JH_z(z,t), \quad z \in \mathbb{R}^{2N}$$

where J is the standard symplectic matrix. The problem is delicate since the Morse indices of the corresponding critical points are infinite. The existence of nontrivial solutions of asymptotically linear sytems is considered in Theorems 6.5 and 6.6. Another application is the existence of at least 2^{2N} periodic solutions when the Hamiltonian is periodic in all variables and when all the periodic solutions are nondegenerate.

We refer to [13] and [14] for surveys on Morse theory with historical remarks. The homotopy index, introduced by C. Conley, is a generalization of the Morse index. We refer to the monograph by Rybakowski [44], and, for a Morse theory based on the Conley index, to Benci [9]. A useful survey of Morse theory in the context of nonsmooth critical point theory is due to Degiovanni [23]. Morse theory on infinite-dimensional manifolds for functions with infinite Morse index is due to Witten and Floer. Here we refer the reader to [34, 40, 45, 46].

2 Global theory

2.1 Preliminaries

Let H_* denote a homology theory with coefficients in a field $\mathbb{F}$. Thus H_* associates to a pair (X, Y) of topological spaces $Y \subset X$ a sequence of $\mathbb{F}$-vector spaces $H_n(X, Y)$, $n \in \mathbb{Z}$, and to a continuous map $f : (X, Y) \to (X', Y')$ a sequence of homomorphisms $f_* : H_*(X, Y) \to H_*(X', Y')$ satisfying the Eilenberg-Steenrod axioms. In particular, given a triple (X, Y, Z) of spaces $Z \subset Y \subset X$ there is a long exact sequence

$$\ldots \xrightarrow{j_{n+1}} H_{n+1}(X,Y) \xrightarrow{\partial_{n+1}} H_n(Y,Z) \xrightarrow{i_n} H_n(X,Z) \xrightarrow{j_n} H_n(X,Y) \xrightarrow{\partial_n} H_{n-1}(Y,Z) \xrightarrow{i_{n-1}} \ldots$$

Given two homomorphisms $V_1 \overset{\varphi_1}{\to} V_2 \overset{\varphi_2}{\to} V_3$ with image $\varphi_1 = \ker\varphi_2$ we have $\dim V_2 = \operatorname{rank}\varphi_1 + \operatorname{rank}\varphi_2$. It follows that

$$\begin{aligned} \dim H_n(X,Z) &= \operatorname{rank} i_n + \operatorname{rank} j_n \\ &= \dim H_n(Y,Z) + \dim H_n(X,Y) - \operatorname{rank}\partial_{n+1} - \operatorname{rank}\partial_n \end{aligned} \tag{2.1}$$

holds for $n \in \mathbb{Z}$. We define the formal series

$$P(t; X, Y) := \sum_{n=0}^{\infty} [\dim H_n(X,Y)] t^n$$

and

$$Q(t; X, Y, Z) := \sum_{n=0}^{\infty} [\operatorname{rank}\partial_{n+1}] t^n.$$

It follows from (2.1) and $\partial_0 = 0$ that:

$$P(t; X, Y) + P(t; Y, Z) = P(t; X, Z) + (1+t) Q(t; X, Y, Z). \tag{2.2}$$

This is correct even when some of the coefficients are infinite so that the series lie in $\overline{\mathbb{N}}_0[[t]]$, $\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$.

Proposition 2.1 *Given a sequence $X_0 \subset X_1 \subset \ldots \subset X_m$ of topological spaces there exists a series $Q(t) \in \overline{\mathbb{N}}_0[[t]]$ such that*

$$\sum_{k=1}^{m} P(t; X_k, X_{k-1}) = P(t; X_m, X_0) + (1+t) Q(t).$$

Proof. By (2.2) the result is true for $m = 2$. If it holds for $m - 1 \geq 2$ then

$$\begin{aligned} \sum_{k=1}^{m} P(t; X_k, X_{k-1}) &= P(t; X_m, X_{m-1}) + P(t; X_{m-1}, X_0) + (1+t) Q_{m-1}(t) \\ &= P(t; X_m, X_0) + (1+t) Q(t; X_m, X_{m-1}, X_0) + (1+t) Q_{m-1}(t) \\ &= P(t; X_m, X_0) + (1+t) Q_m(t). \end{aligned}$$

□

Remark 2.2 *Let $m_n, p_n \in \mathbb{N}_0$ be given for $n \geq 0$, and set $M(t) := \sum_{n=0}^{\infty} m_n t^n$, $P(t) := \sum_{n=0}^{\infty} p_n t^t$. Then the following are equivalent:*

(i) There exists a series $Q(t) \in \mathbb{N}_0[[t]]$ with $M(t) = P(t) + (1+t)Q(t)$

(ii) $m_n - m_{n-1} + m_{n-2} - + \ldots + (-1)^n m_0 \geq p_n - p_{n-1} + p_{n-2} - + \ldots + (-1)^n p_0$ holds for every $n \in \mathbb{N}_0$

In fact, if (i) holds with $Q(t) = \sum_{n=0}^{\infty} q_n t^n$ we have $m_n = p_n + q_n + q_{n-1}$ where $q_{-1} := 0$, hence

$$\sum_{i=0}^{n}(-1)^{n-i} m_i = \sum_{i=0}^{n}(-1)^{n-i}(p_i + q_i + q_{i-1}) = \sum_{i=0}^{n}(-1)^{n-i} p_i + q_n \geq \sum_{i=0}^{n}(-1)^{n-i} p_i.$$

On the other hand, if (ii) holds we define $q_{-1} := 0$,

$$q_n := \sum_{i=0}^{n}(-1)^{n-i} m_i - \sum_{i=0}^{n}(-1)^{n-i} p_i \in \mathbb{N}_0, \qquad n \geq 0,$$

and obtain $q_n + q_{n-1} = m_n - p_n$ for all $n \geq 0$. This yields (i).

Clearly, (i) and (ii) imply $m_n \geq p_n$ for all $n \in \mathbb{N}_0$. If $M(t) = \sum_{n=0}^{N} m_n t^n$, $P(t) = \sum_{n=0}^{N'} p_n t^n$ are polynomials and (i), (ii) hold then $N = N'$ and applying (ii) for $n = N$, $N+1$ yields the equality:

$$\sum_{i=0}^{N}(-1)^{N-i} m_i = \sum_{i=0}^{N}(-1)^{N-i} p_i.$$

As a consequence of these observations we obtain:

Corollary 2.3 *Consider a sequence $X_0 \subset X_1 \subset \ldots \subset X_m$ of topological spaces such that all Betti numbers $\dim H_n(X_k, X_{k-1})$ are finite, and set*

$$m_n := \sum_{k=1}^{m} \dim H_n(X_k, X_{k-1}), \qquad p_n := \dim H_n(X_m, X_0).$$

Then

$$\sum_{i=0}^{n}(-1)^{n-i} m_i \geq \sum_{i=0}^{n}(-1)^{n-i} p_i$$

holds for all $n \geq 0$. In particular, $m_n \geq p_n$ holds for all $n \in \mathbb{N}_0$. Moreover, if $m_n = 0$ for $n > N$ then $p_n = 0$ for $n > N$ and

$$\sum_{i=0}^{N}(-1)^{N-i} m_i = \sum_{i=0}^{N}(-1)^{N-i} p_i.$$

Remark 2.4 *In this section we have only used the exact sequence of a triple for H_*. We may also work with cohomology instead of homology.*

2.2 The Morse inequalities

Let X be a topological space, $f : X \to \mathbb{R}$ be continuous, $K \subset X$ closed. In the differentiable setting K will be the set of critical points of f and $f(K)$ the set of critical values. For $c \in \mathbb{R}$ we set $f^c := \{x \in X : f(x) \leq c\}$ and $K_c := \{x \in K : f(x) = c\}$. For $c \leq d$ we shall also use the notation $f_c^d := \{x \in X : c \leq f(x) \leq d\}$. Let H_* be a homology theory. For $c \in \mathbb{R}$ and $S \subset f^{-1}(c)$ we define

$$C_n(f, S) := H_n(f^c, f^c \setminus S), \quad n \in \mathbb{Z}.$$

For $x \in f^{-1}(c)$ we set

$$C_n(f, x) := C_n(f, \{x\}) = H_n(f^c, f^c \setminus \{x\}), \quad n \in \mathbb{Z}.$$

Lemma 2.5 *If $S_1, S_2 \subset f^{-1}(c)$ are closed and disjoint then $C_n(f, S_1 \cup S_2) = C_n(f, S_1) \oplus C_n(f, S_2)$.*

Proof. This follows from the relative Mayer-Vietoris sequence of the triad $(f^c; f^c \setminus S_1, f^c \setminus S_2)$. Observe that $(f^c \setminus S_1) \cap (f^c \setminus S_2) = f^c \setminus (S_1 \cup S_2)$ and $(f^c \setminus S_1) \cup (f^c \setminus S_2) = f^c$. Thus we have an exact sequence

$$\begin{array}{ccccccc} H_{n+1}(f^c, f^c) & \to & H_n(f^c, f^c \setminus (S_1 \cup S_2)) & \to & H_n(f^c, f^c \setminus S_1) \oplus H_n(f^c, f^c \setminus S_2) & \to & H_n(f^c, f^c) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

□

Now we fix two real numbers $a < b$ and require:

(A1) $f(K) \cap [a, b] = \{c_1, \dots, c_k\}$ is finite and $c_0 := a < c_1 < c_2 < \dots < c_k < c_{k+1} := b$

(A2) $H_*(f^{c_{j+1}} \setminus K_{c_j+1}, f^{c_j}) = 0$ for $j = 0, \dots, k$, $n \in \mathbb{Z}$.

If one wants to prove the existence of one or many critical points of f using Morse theory one can assume (A1) to hold because otherwise one has already infinitely many critical values. Condition (A2) is equivalent to:

(A3) The inclusion $i_j : f^{c_j} \hookrightarrow f^{c_{j+1}} \setminus K_{c_j+1}$ induces an isomorphism $i_{j*} : H_*(f^{c_j}) \to H_*(f^{c_{j+1}} \setminus K_{c_j+1})$ in homology for all $j = 0, \dots, k$.

In classical Morse theory, f^{c_j} will in fact be a strong deformation retract of $f^{c_{j+1}} \setminus K_{c_j+1}$. Recall that $Z \subset Y$ is a strong deformation retract of Y if there exists a continuous map $h : [0, 1] \times Y \to Y$ such that $h(0, x) = x$ and $h(1, x) \in Z$ for all $x \in Y$, and $h(t, x) = x$ for all $x \in Z$, $0 \le t \le 1$. h deforms Y into Z keeping Z fixed. This implies that $Z \hookrightarrow Y$ is a homotopy equivalence, hence it induces an isomorphism in homology.

Now for $n \ge 0$ we define

$$m_n := \sum_{j=1}^{k} \dim C_n(f, K_{c_j}) \in \overline{\mathbb{N}}_0 \quad \text{and} \quad p_n := \dim H_n(f^b, f^a) \in \overline{\mathbb{N}}_0.$$

We also set

$$M_f(t; a, b) := \sum_{n=0}^{\infty} m_n t^n \in \overline{\mathbb{N}}_0[[t]] \quad \text{and} \quad P_f(t; a, b) := \sum_{n=0}^{\infty} p_n t^n \in \overline{\mathbb{N}}_0[[t]]$$

If these are polynomials with finite coefficients then $M_f(t; a, b)$ is called Morse polynomial, $P_f(t; a, b)$ Poincaré polynomial.

Theorem 2.6 *If (A1), (A2) hold then there exists a series $Q(t) \in \overline{\mathbb{N}}_0[[t]]$ such that*

$$M_f(t; a, b) = P_f(t; a, b) + (1 + t)Q(t).$$

If in addition all coefficients m_n, p_n of $M_f(t; a, b)$, $P_f(t; a, b)$ are finite then the following Morse inequalities hold:

$$\sum_{i=0}^{n} (-1)^{n-i} m_i \ge \sum_{i=0}^{n} (-1)^{n-i} p_i, \quad \text{all } n \ge 0.$$

Consequently, $m_n \ge p_n$ for all $n \ge 0$ and, if $M_f(t; a, b) \in \mathbb{N}_0[t]$ is a polynomial of degree N so is $P_f(t; a, b)$. In that case, the Morse equality

$$\sum_{i=0}^{N} (-1)^{N-i} m_i = \sum_{i=0}^{N} (-1)^{N-i} p_i$$

holds.

Proof. The long exact sequence of the triple $(f^{c_j}, f^{c_j} \setminus K_{c_j}, f^{c_j - 1})$ and (A2) yield:

$$C_n(f, K_{c_j}) = H_n(f^{c_j}, f^{c_j} \setminus K_{c_j}) = H_n(f^{c_j}, f^{c_j - 1}) \quad \text{for } j = 1, \dots, k+1, \ n \in \mathbb{Z}.$$

The theorem follows from Proposition 2.1 and Corollary 2.3 applied to $X_j = f^{c_j}, j = 0, \dots, k+1$. □

What we presented so far is just elementary algebraic topology. Analysis enters when proving (A2) for certain (classes of) maps $f : X \to \mathbb{R}$. As mentioned above, (A1) is usually assumed to hold. We begin with a simple and classical situation.

Proposition 2.7 *Let X be a smooth closed (i.e. compact without boundary) riemannian manifold, $f \in C^2(X, \mathbb{R})$, and let $a < b$ be regular values of f. Let $K := \{x \in X : f'(x) = 0\}$ be the set of critical points of f and assume that $K \cap f_a^b$ is finite. Then (A1) and (A2) hold.*

In fact, (A1) holds trivially true. (A2) follows from a stronger result.

Proposition 2.8 *In the situation of Proposition 2.7 let $c, d \in (a, b)$, $c < d$, be such that $f(K) \cap (c, d) = \emptyset$. Then f^c is a strong deformation retract of $f^d \setminus K_d$.*

The proof of Proposition 2.8 uses the negative gradient flow associated to f. If $\langle \cdot , \cdot \rangle$ denotes the riemannian metric on X then $\nabla f(x) \in T_x X$ is defined by $\langle \nabla f(x), v \rangle = f'(x)v$ for all $v \in T_x X$. Since f is C^2, $\nabla f : X \to TX$ is a C^1-vector field and induces a flow φ_f on X defined by

$$\begin{cases} \dfrac{d}{dt}\varphi_f(t, x) = -\nabla f(\varphi_f(t, x)) \\ \varphi_f(0, x) = x. \end{cases}$$

We only need the induced semiflow on $Y := X \setminus K$ which we simply denote by $\varphi : [0, \infty) \times Y \to Y$. We also write $\varphi^t(x) := \varphi(t, x)$ and denote the ϵ-neighborhood of K by $U_\epsilon(K)$.

Lemma 2.9 *φ has the following properties:*

(φ1) *For every $\epsilon > 0$ there exists $\delta > 0$ such that for $x \in Y \cap f_a^b$, $t > 0$ there holds: If $\varphi^s(x) \in f_a^b \setminus U_\epsilon(K)$ for all $0 \le s \le t$ then $f(x) - f(\varphi^t(x)) \ge \delta d(x, \varphi^t(x)) > 0$.*

(φ2) *If $f(\varphi^t(x)) \ge a$ for all $t \ge 0$ then the orbit $\{\varphi^t(x) : t \ge 0\}$ is relatively compact in X.*

In (φ1), $d(x, y) = \inf_\gamma \int_0^1 \|\dot\gamma(t)\| dt$ denotes the distance in X; the infimum extends over all C^1-paths $\gamma : [0, 1] \to X$ with $\gamma(0) = x$, $\gamma(1) = y$. According to (φ1), for every $x \in Y \cap f_a^b$ the level $f(\varphi^t(x))$ strictly decreases as a function of t, as long as $\varphi^t(x) \in Y \cap f_a^b$. Moreover, if $\varphi^t(x)$ stays uniformly away from the set K, the difference quotient $\big(f(x) - f(\varphi^t(x))\big)/d(x, \varphi^t(x))$ is bounded away from 0.

Proof. (φ2) is clear because X is compact. In order to prove (φ1) fix $\epsilon > 0$ and set

$$\delta := \inf\{\|\nabla f(x)\| : \ x \in f_a^b \setminus U_\epsilon(K)\} > 0.$$

Then we have for x, t as in (φ1):

$$\begin{aligned} f(x) - f(\varphi^t(x)) &= -\int_0^t \frac{d}{ds} f(\varphi^s(x)) ds = \int_0^t \|\nabla f(\varphi^s(x))\|^2 ds \\ &\ge \delta \int_0^t \|\nabla f(\varphi^s(x))\| ds = \delta \int_0^t \|\frac{d}{ds}\varphi^s(x)\| ds \ge \delta d(\varphi^t(x), x). \end{aligned}$$

□

For the proof of Proposition 2.8 we only use the properties ($\varphi 1$), ($\varphi 2$). We need the map

$$\tau : f^d \setminus K_d \to [0, \infty], \quad \tau(x) := \inf\{t \geq 0 : f(\varphi^t(x)) \leq c\}.$$

Here $\inf \emptyset = \infty$, so $\tau(x) = \infty$ if and only if $f(\varphi^t(x)) > c$ for all $t \geq 0$. Clearly $\tau(x) = 0$ if and only if $x \in f^c$.

Lemma 2.10 *If $\tau(x) = \infty$ then $\varphi^t(x) \to \overline{x} \in K_c$ as $t \to \infty$.*

Proof. By ($\varphi 2$) the ω-limit set

$$\omega(x) := \bigcap_{t \geq 0} \mathrm{clos}\{\varphi^s(x) : s \geq t\} = \{y \in X : \text{ there exists } t_n \to \infty \text{ with } \varphi^{t_n}(x) \to y\}$$

is a compact connected nonempty subset of X. ($\varphi 1$) implies that $\omega(x) \subset K$, hence $\omega(x) \subset K_c$ because $f(K) \cap (c, d) = \emptyset$ by assumption. Finally, since K_c is finite we obtain $\omega(x) = \{\overline{x}\}$ for some $\overline{x} \in K_c$. □

Lemma 2.11 *τ is continuous.*

Proof. Fix $x \in f^d \setminus K_d$ and $t < \tau(x)$. Then $f(\varphi^t(x)) > c$, hence there exists a neighborhood N of x with $f(\varphi^t(y)) > c$ for every $y \in N$. This implies $\tau(y) > t$ for $y \in N$, so τ is lower semi-continuous. Analogously one shows that τ is upper semi-continuous. □

Proof of Proposition 2.8. Clearly, the flow φ deforms any $x \in Z := f^d \setminus K_d$ within the time $\tau(x)$ to a point in f^c. In order to define the deformation $h : [0, 1] \times Z \to Z$ required in Proposition 2.8 we just need to rescale the interval $[0, 1]$ to $[0, \tau(x)]$. This is achieved, for instance, by the map

$$\chi(s, t) := \begin{cases} \dfrac{st}{1 + t - st} & \text{for } 0 \leq s \leq 1,\ 0 \leq t < \infty; \\ \dfrac{s}{1 - s} & \text{for } 0 \leq s < 1,\ t = \infty. \end{cases}$$

For fixed $t < \infty$ we have a reparametrization $\chi(\cdot, t) : [0, 1] \to [0, t]$, while for $t = \infty$ we have a reparametrization $\chi(\cdot, \infty) : [0, 1) \to [0, \infty)$. Clearly χ is continuous.

Now we define

$$h : [0, 1] \times Z \to Z, \quad h(s, x) := \begin{cases} \lim\limits_{t \to \infty} \varphi^t(x) & \text{if } s = 1,\ \tau(x) = \infty; \\ \varphi(\chi(s, \tau(x)), x) & \text{else.} \end{cases}$$

Then $h(0,x) = \varphi(0,x) = x$, $h(1,x) = \varphi(\tau(x),x) \in f^c$ if $\tau(x) < \infty$ and $h(1,x) = \lim_{t\to\infty} \varphi^t(x) \in K_c \subset f^c$ if $\tau(x) = \infty$. Moreover, if $x \in f^c$ then $\tau(x) = 0$ and $h(t,x) = \varphi(0,x) = x$ for all $t \in [0,1]$.

It remains to prove that h is continuous. Since φ, χ, τ are continuous we only need to consider the continuity at points $(1,x)$ with $\tau(x) = \infty$. We first show that $f \circ h$ is continuous at $(1,x)$. For $\varepsilon > 0$ there exists $t_\varepsilon \geq 0$ with $f(\varphi^{t_\varepsilon}(x)) < c+\varepsilon$. There also exists a neighborhood N_ε of x with $f(\varphi^{t_\varepsilon}(y)) < c+\varepsilon$ for all $y \in N_\varepsilon$, hence $c < f(\varphi^t(y)) < c+\varepsilon$ for all $y \in N_\varepsilon$, $t \in [t_\varepsilon, \tau(y)]$. This implies $c \leq f(h(s,y)) < c+\varepsilon$ for all $y \in N_\varepsilon$, all $s \geq \chi(\cdot,\tau(y))^{-1}(t_\varepsilon) =: s_{\varepsilon,y}$. By the continuity of τ we may assume that $\tau(y) \geq t_\varepsilon + 1$ for all $y \in N_\varepsilon$. This implies that $s_\varepsilon := \sup_{y\in N_\varepsilon} s_{\varepsilon,y} < 1$, and $c \leq f(h(s,y)) < c+\varepsilon$ for all $s \in [s_\varepsilon, 1]$, $y \in N_\varepsilon$. Thus $f \circ h$ is continuous.

Set $\bar{x} := h(1,x)$ and suppose there exist sequences $s_n \to 1$, $x_n \to x$ such that $h(s_n,x_n) \notin \overline{U}_{2\varepsilon}(\bar{x})$ for some $\varepsilon > 0$. Since $\varphi^k(x) \to \bar{x}$ as $k \to \infty$ and since φ is continuous, there exists a subsequence x_{n_k} with $\varphi^k(x_{n_k}) \to \bar{x}$. Setting $t_{n_k} := \chi(\cdot,\tau(x_{n_k}))^{-1}(k)$ we have $h(t_{n_k}, x_{n_k}) \to \bar{x}$ and $t_{n_k} \to 1$. Thus, after passing to a subsequence, we may assume that $h(t_n,x_n) \to \bar{x}$ for some sequence $t_n \to 1$. We may also assume that $U_{3\varepsilon}(\bar{x}) \cap K = \{\bar{x}\}$. Then between the times s_n and t_n, the orbit $\varphi^t(x)$ passes through $\overline{U}_{2\varepsilon}(\bar{x}) \setminus \overline{U}_\varepsilon(\bar{x}) \subset f_a^b \setminus U_\varepsilon(K)$, so ($\varphi$1) yields $\delta > 0$ such that $|f(h(s_n,x_n)) - f(h(t_n,x_n))| \geq \delta\varepsilon/2$ for all $n \in \mathbb{N}$. On the other hand, since $f \circ h$ is continuous we obtain $f(h(s_n,x_n)) - f(h(t_n,x_n)) \to 0$, a contradiction. $\square$

If, in the smooth case of Proposition 2.7 all critical points in $K \cap f_a^b$ are nondegenerate then $\dim C_n(f^c, f^c \setminus K_c)$ is precisely the number of critical points in K_c with Morse index n. This will be proved in section 3.2 below. Here x is a nondegenerate critical point of f if the hessian $f''(x) : T_xX \times T_xX \to \mathbb{R}$ is a nondegenerate quadratic form. The Morse index of x is the maximal dimension of a subspace of T_xX on which $f''(x)$ is negative definite. Combining Theorem 3.6 with Theorem 2.6 and Proposition 2.7 we obtain

Theorem 2.12 *Let X be a smooth closed riemannian manifold, $f \in C^2(X,\mathbb{R})$, and let $a < b$ be regular values of f. Let $K = \{x \in X : f'(x) = 0\}$ and suppose that all critical points in $K \cap f_a^b$ are nondegenerate. For $i \in \mathbb{N}_0$ let $m_i \in \mathbb{N}_0$ be the number of critical points in $K \cap f_a^b$ with Morse index i, and let $p_i := \dim H_i(f^b, f^a)$. Then there exists a polynomial $Q(t) \in \mathbb{N}_0[t]$ such that*

$$\sum_{i=0}^{\dim X} m_i t^i = \sum_{i=0}^{\dim X} p_i t^i + (1+t)Q(t).$$

Equivalently, the Morse inequalities

$$\sum_{i=0}^{n} (-1)^{n-i} m_i \geq \sum_{i=0}^{n} (-1)^{n-i} p_i, \quad n \geq 0$$

hold.

In the special case $a < \min f$, $b > \max f$ we have that $p_i = \dim H_i(X)$ is the i-th Betti number of X and m_i is the number of all critical points of f with Morse index i. Then we obtain the classical Morse inequalities.

In applications to boundary value problems for ordinary or partial differential equations, X is an infinite-dimensional Hilbert space or Hilbert manifold and $f : X \to \mathbb{R}$ is often only of class C^1. In that case one needs a replacement for the negative gradient flow. This is being achieved by considering pseudo-gradient vector fields. A vector $v \in X$ (or $v \in T_xX$) is said to be a pseudo-gradient vector for f at x if the following two conditions are satisfied:

(pg1) $\|v\| < 2\|f'(x)\|$

(pg2) $f'(x)v > \frac{1}{2}\|f'(x)\|^2$

A pseudo-gradient vector field for f on $Y \subset X$ is a locally Lipschitz continuous vector field $V : Y \to TX$ such that $V(x)$ is a pseudo-gradient vector for f at x. Using partitions of unity it is easy to construct a pseudo-gradient vector field for f on $X \setminus K$; see [17, Lemma I.3.1]. The conditions (φ1), (φ2) do not hold in general, however. They do hold if the following Palais-Smale condition is satisfied for $c \in \mathbb{R}$.

(PS)$_c$ Every Palais-Smale sequence x_n in X, i.e. a sequence such that $f'(x_n) \to 0$ and $f(x_n) \to c$, has a convergent subsequence.

Proposition 2.13 *Let X be a complete C^2-Hilbert manifold (without boundary), $f : X \to \mathbb{R}$ be C^1 and let K be the set of critical points of f. Let $a < b$ be given and suppose that the Palais-Smale condition (PS)$_c$ holds for every $c \in [a, b]$. Then there exists a flow φ on $Y := X \setminus K$ with the properties (φ1), (φ2) from Lemma 2.9.*

Proof. One constructs a pseudo-gradient vector field V for f on Y and takes φ to be the semi-flow induced by $-V$. The conditions (φ1), (φ2) follow easily since $\|f'(x)\|$ is bounded away from 0 for $x \in f_a^b \setminus U_\varepsilon(K)$. We cheated a bit because φ is not defined on $Y \times [0, \infty)$, in general. This can be remedied however by considering the vector field $-\chi V$ with an appropriately chosen cut-off function $\chi : X \to [0, 1]$. Alternatively, one may rewrite the proof of Proposition 2.8 for not globally defined semiflows. □

As a consequence we obtain the Morse inequalities also in the infinite-dimensional setting. In the setting of Proposition 2.13 we call a critical point x of f topologically nondegenerate with Morse index $\mu \in \mathbb{N}_0$ if it is an isolated critical point and $\dim C_k(f, x) = \delta_{k,\mu}$.

Theorem 2.14 *Let X be a C^2-Hilbert manifold, $f : X \to \mathbb{R}$ be C^1 and let K be the set of critical points of f. Let $a < b$ be given and suppose that the Palais-Smale condition (PS)$_c$ holds for every $c \in [a, b]$. Suppose moreover that $f(K) \cap [a, b] = \{c_1 < \cdots < c_k\} \subset (a, b)$ is finite. Then the Morse inequalities from Theorem 2.6 hold. If in addition all critical points in f_a^b are topologically nondegenerate and have finite Morse index then the Morse inequalities from Theorem 2.12 hold.*

If f is C^2 then the concepts of nondegeneracy and Morse index are as in the finite-dimensional setting, and a nondegenerate critical point is topologically non-degenerate; see Definition 3.3 and Theorem 3.6.

Remark 2.15 *For a number of applications it would be useful to develop Morse theory on Banach manifolds. This is rather delicate, however. For instance, if X is a Banach space and $f : X \to \mathbb{R}$ is C^2, then the existence of a nondegenerate critical point of f with finite Morse index implies the existence of an equivalent Hilbert space structure on X. Extensions of Morse theory to the Banach space setting are still a topic of research; see [16, 19, 41, 52, 53].*

Remark 2.16 *The Morse theory developed here can be refined to localize critical points.*

a) Suppose the semi-flow associated to a pseudo-gradient vector field leaves a subset $Z \subset Y$ positively invariant, that is if $x \in Z$ then $\varphi^t(x) \in Z$ for all $t \geq 0$. In order to find critical points in Z one may replace X by Z and K by $K \cap Z$. The conditions ($\varphi 1$), ($\varphi 2$) continue to hold so that one obtains the Morse inequalities constrained to Z. It is important to note however that the coefficients in the Morse and the Poincaré polynomials depend on Z. This idea can be used for instance, to find positive (or negative) solutions of semilinear elliptic boundary value problems, that is, solutions lying in the cone of positive functions – provided one can construct a flow φ as above leaving this cone positively invariant.

b) One can also localize critical points outside of a positive invariant set Z. It can also be used to find sign-changing solutions of elliptic boundary value problems, that is solutions lying outside of the cone of positive or negative functions; see [18] for such an application. A first idea is to use the inverse flow $\varphi_-(t, x) = \varphi(-t, x)$ corresponding to $-f$. Observe that $X \setminus Z$ is positive invariant with respect to φ_- if Z is positive invariant with respect to φ. In the infinite-dimensional setting this does not work so easily because the theory developed so far yields nontrivial results only if the Morse indices are finite. If X is an infinite-dimensional manifold and $x \in X$ is a critical point of f of finite Morse index (and finite nullity) then it has infinite Morse index considered as a critical point of $-f$. Instead one can set up a relative Morse theory replacing X by the pair (X, Z) and K by $K \cap (X \setminus Z)$. The coefficients m_n, p_n are now defined as $m_n = \sum_{j=1}^k \dim C_n(f, K_{c_j} \cap (X \setminus Z))$ and $p_n = \dim H_n(f^b, f^a \cup Z)$.

3 Local theory

3.1 Morse lemma

The Morse lemma is the basic tool for the computation of the critical groups of a nondegenerate critical point. For degenerate isolated critical points, the splitting theorem gives the appropriate representation. Since the theory is local, we consider a Hilbert space E. We require in this section

(M) U is an open neighborhood of 0 in the Hilbert space E, 0 is the only critical point of $f \in \mathcal{C}^2(U, \mathbb{R})$, $L = f''(0)$ is invertible or 0 is an isolated point of $\sigma(L)$.

Here, using the scalar product of E, the Hessian $f''(0) : E \times E \to \mathbb{R}$ of f at 0 corresponds to a linear map $L : E \to E$. By abuse of notation we write $f''(0)$ for both maps.

Theorem 3.1 (Morse Lemma.) *Let L be invertible. Then there exists an open ball B_δ and local diffeomorphism $g : B_\delta \to E$ such that $g(0) = 0$ and, on B_δ,*

$$f \circ g(u) = \frac{1}{2}(Lu, u).$$

If 0 is an isolated point of $\sigma(L)$, E is the orthogonal sum of $R(L)$, the range of L, and $N(L)$, the kernel of L. Let $u = v + w$ be the corresponding decomposition of $u \in E$.

Theorem 3.2 (Splitting Theorem.) *Let 0 be an isolated point of $\sigma(L)$. Then there exists an open ball B_δ, a local homeomorphism $g : B_\delta \to E$ such that $g(0) = 0$ and a $\mathcal{C}^1$ mapping $h : B_\delta \cap N(L) \to R(L)$ such that, on B_δ,*

$$f \circ g(v + w) = \frac{1}{2}(Lv, v) + f(h(w) + w) = \frac{1}{2}(Lv, v) + \hat{f}(w).$$

For the proofs of Theorems 3.1 and 3.2, we refer to [17] and to [39].

3.2 Critical groups

In this section, we denote by U an open subset of the Hilbert space E. Let us recall a definition of section 2.2 in this setting.

Definition 3.3 *Let x be an isolated critical point of $f \in \mathcal{C}^1(U, \mathbb{R})$. The critical groups of x are defined by*

$$C_n(f, x) = H_n(f^c, f^c \backslash \{x\}), n \in \mathbb{Z},$$

where $c = f(x)$.

Remark 3.4 *By excision, the critical groups depend only on the restriction of f to an arbitrary neighborhood of x in U. The critical groups of a critical point of $f \in \mathcal{C}^2(X,\mathbb{R})$, where X is a $\mathcal{C}^2$-Hilbert manifold are defined in the same way.*

Definition 3.5 *Let x be a critical point of $f \in \mathcal{C}^2(U,\mathbb{R})$. The Morse index of x is the supremum of the dimensions of the subspaces of E on which $f''(x)$ is negative definite. The critical point x is nondegenerate if $f''(x)$ is invertible. The nullity of x is the dimension of the kernel of $f''(x)$.*

Theorem 3.6 *Let x be a nondegenerate critical point of $f \in \mathcal{C}^2(U,\mathbb{R})$ with finite Morse index m. Then*

$$\dim C_n(f,x) = \delta_n^m.$$

Proof. We can assume that $x = 0$ and $f(x) = 0$. By Theorem 3.1, there exists a local diffeomorphism $g : B_\delta \to E$ such that $g(0) = 0$ and, on B_δ,

$$f \circ g(u) = \frac{1}{2}(Lu,u) := \psi(u).$$

We have, for $m \geq 1$,

$$\begin{aligned} C_n(f,0) &= H_n(f^0 \cap g(B_\delta), f^0 \cap g(B_\delta)\backslash\{0\}) \\ &\cong H_n(\psi^0 \cap B_\delta, \psi^0 \cap B_\delta\backslash\{0\}) \\ &\cong H_n(B^m, S^{m-1}) \end{aligned}$$

and

$$\dim H_n(B^m, S^{m-1}) = \delta_n^m.$$

We have also, for $m = 0$,

$$C_n(f,0) \cong H_n(\{0\},\emptyset)$$

and

$$\dim H_n(\{0\},\emptyset\} = \delta_n^0.$$

□

Theorem 3.7 (Shifting Theorem.) *Let U be an open neighborhood of 0 in the Hilbert space E. Assume that 0 is the only critical point of $f \in \mathcal{C}^2(U,\mathbb{R})$, that the Palais-Smale condition is satisfied over a closed ball in U and that the Morse index m of 0 is finite. Then*

$$C_n(f,0) \cong C_{n-m}(\hat{f},0)$$

where $\hat{f}$ is defined in Theorem 3.2.

For the proof we refer to Theorem 8.4 in [39].

Corollary 3.8 *Assume moreover that the nullity k of 0 is finite. Then*

a) $C_n(f, 0) \cong 0$ *for* $n \leq m - 1$ *and* $n \geq m + k + 1$*;*

b) 0 *is a local minimum of* $\hat{f}$ *iff* $\dim C_m(f, 0) = \delta_n^m$*;*

c) 0 *is a local maximum of* $\hat{f}$ *iff* $\dim C_m(f, 0) = \delta_n^{m+k}$*.*

For the proof see Corollary 8.4 in [39].

4 Applications

4.1 A three critical points theorem

Let E be a Hilbert space and $f \in \mathcal{C}^1(E, \mathbb{R})$. As in section 2.2, f satisfies the $(\mathrm{PS})_c$ condition, $c \in \mathbb{R}$, if every sequence (x_n) in X with $f'(x_n) \to 0$ and $f(x_n) \to c$ has a convergent subsequence. The function f satisfies the (PS) condition if, for every $c \in \mathbb{R}$, the $(\mathrm{PS})_c$ condition is satisfied.

Theorem 4.1 *Let $f \in \mathcal{C}^2(E, \mathbb{R})$ be bounded from below. Assume that f satisfies the (PS)-condition and that x_1 is a nondegenerate nonminimum critical point of f with finite Morse index. Then f has at least three critical points.*

Proof. Since f is bounded from below and satisfies (PS), there exists a minimizer x_0 of f on E. Let us assume that x_0 and x_1 are the only critical points of f. Corollary 3.8 and Theorem 3.6 yield

$$\dim C_n(f, x_0) = \delta_n^0, \quad \dim C_n(f, x_1) = \delta_n^m,$$

where m is the Morse index of x_1. Theorem 2.14, applied to $a = f(x_0) - 1$ and $b = f(x_1) + 1$ yields a polynomial $Q \in \mathbb{N}_0[t]$

$$1 + t^m = 1 + (1 + t)Q(t),$$

a contradiction. We have used the fact that f^b is a deformation retract of E so that, for $n \in \mathbb{Z}$,

$$H_n(f^b, f^a) = H_n(E, \emptyset) = H_n(E).$$

$\square$

Let us consider the Dirichlet problem

$$(4.1) \qquad \begin{cases} -\Delta u = g(u), & \text{in } \Omega, \\ \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. We assume that

(g_1) $g \in \mathcal{C}^1(\mathbb{R})$ and

$$|g'(t)| \leq c_1(1 + |t|^{p-2}), 2 \leq p < 2^*,$$

with $2^* = +\infty$ if $N = 2$, $2^* = 2N/(N-2)$ if $N \geq 3$.

Let $\lambda_1 < \lambda_2 \leq \ldots$ be the eigenvalues of $-\Delta$ with the Dirichlet boundary condition on Ω. We assume also that

(g_2) $G(t) \leq c_2(1 + t^2)$, $c_2 < \lambda_1/2$, where $G(t) = \int_0^t g(s)ds$,

(g_3) $g(0) = 0$, $\lambda_j < g'(0) < \lambda_{j+1}$, $j \geq 1$.

It follows from (g_1) that the solutions of (4.1) are the critical points of the $\mathcal{C}^2$-functional

$$f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G(u)dx,$$

defined on $H_0^1(\Omega)$. The space $H_0^1(\Omega)$ is the subspace of functions of

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) : \frac{\partial u}{\partial x_k} \in L^2(\Omega), 1 \leq k \leq N \right\}$$

satisfying the Dirichlet boundary condition in the sense of traces.

Theorem 4.2 *Under assumptions (g_{1-2-3}), problem (4.1) has at least 3 solutions.*

Proof. It follows from (g_2) that

$$f(u) \to +\infty \quad \text{as } \|u\| = \|\nabla u\|_{L^2} \to \infty, \tag{4.2}$$

on $H_0^1(\Omega)$. By Rellich's theorem, the imbedding $H_0^1(\Omega) \subset L^p(\Omega)$ is compact for $p < 2^*$. Using (g_1), it is then not difficult to verify the (PS)-condition. In particular, by (4.2), there exists a minimizer x_0. By assumption (g_3), 0 is a nondegenerate critical point of f with Morse index $j \geq 1$. In particular $x_0 \neq 0$. By Theorem 4.1, f has at least 3 critical points. □

4.2 Asymptotically linear problems

We consider again problem (4.1) under the assumptions (g_1) and

(g_4) $g(0) = 0$, $\lambda_j < g'(0) < \lambda_{j+1}$, $j \geq 0$, where $\lambda_0 = -\infty$,

(g_5) $g(t) = \lambda t + o(t)$, $|t| \to +\infty$, $\lambda_k < \lambda < \lambda_{k+1}$, $0 \leq k \neq j$.

Theorem 4.3 *Under assumptions (g_{1-4-5}), problem* (4.1) *has at least 2 solutions.*

Proof. Let us assume that 0 is the only solution and, hence, the only critical point of f. As a consequence of (g_1) and (g_5) the (PS)-condition holds (cf. [17]). Moreover, for $b > 0$ and $a < 0$,

$$\dim H_n(f^b, f^a) = \delta_n^k.$$

By Theorem 3.6,

$$\dim C_n(f, 0) = \delta_n^j.$$

It follows from Theorem 2.14 that $j = k$, contrary to our assumptions. □

More solutions exist if

(g_6) g is odd.

Theorem 4.4 *Under assumptions $(g_{1-4-5-6})$, problem* (4.1) *has at least $m = |j-k|$ pairs $\pm u_1, \ldots, \pm u_m$ of solutions in addition to the trivial solution 0.*

Proof. We sketch the proof in the case $j > k$. Let $G = \{\pm 1\}$ denote the group of 2 elements acting on $E := H^1(\Omega)$ via the antipodal map $u \mapsto -u$. Observe that f is even by (g_6), hence sublevel sets f^a are invariant under G. Moreover, if u is a critical point of f then so is $-u$ because f' is odd: $f'(u) = -f'(-u)$. We consider the Borel cohomology $H_G^*(f^a)$ of the sublevel set f^a for $a < 0$. Since 0 is the only fixed point of the action of G on E and $0 \notin f^a$ for $a < 0$ one has $H_G^*(f^a) \cong H^*(f^a/G; \mathbb{F}_2)$ where $f^a/G = f^a/u \sim -u$ is the quotient space and $\mathbb{F}_2 = \{0, 1\}$ is the field of 2 elements. The cohomology group $H_G^*(f^a)$ is a module over the ring $H_G^*(E \setminus \{0\}) \cong H^*(\mathbb{R}P^\infty) \cong \mathbb{F}_2[w]$.

As a consequence of (g_5), equation (4.1) is asymptotically linear and f is asymptotically quadratic. More precisely, let e_i, $i \in \mathbb{N}$, be an othogonal basis of $H^1(\Omega)$ consisting of eigenfunctions of $-\Delta$ corresponding to the eigenvalues λ_i. Then

$$f(u) \to -\infty \qquad \text{for } u \in E_\infty := \operatorname{span}\{e_1, \ldots, e_k\},\ \|u\| \to \infty$$

and

$$f(u) \to +\infty \qquad \text{for } u \in E_\infty^\perp = \operatorname{span}\{e_i : i \geq k+1\},\ \|u\| \to \infty.$$

by (g_5). Thus for $b \ll 0$ and $R \gg 0$ we have inclusions

$$S_R E_\infty := \{u \in E_\infty : \|u\| = R\} \hookrightarrow f^b \hookrightarrow E \setminus E_\infty^\perp \simeq S_R E_\infty.$$

These are in fact homotopy equivalences which implies that

$$H_G^*(f^b) \cong H_G * (S_R E_\infty) \cong H^*(\mathbb{R}P^k) \cong \mathbb{F}_2[w]/w^{k+1}.$$

On the other hand, $f(u) < 0$ for $u \in E_0 := \operatorname{span}\{e_1, \dots, e_j\}$ with $\|u\| \le r$, $r > 0$ small, by (g_4). Thus for $a < 0$ close to 0 we have the inclusion $S_r E_0 \hookrightarrow f^a \hookrightarrow E \setminus \{0\}$. On the cohomology level this yields homomorphisms

$$\mathbb{F}_2[w] \to H_G^*(f^a) \to \mathbb{F}_2[w]/w^j$$

whose composition is surjective. Consequently, $H_G^i(f^a) \neq 0 = H_G^i(f^b)$ for $i = k+1, \dots, j$, and therefore $H_G^i(f^a, f^b) \neq 0$ for $i = k+1, \dots, j$. If all critical points are nondegenerate we immediately obtain critical points $u_1, \dots, u_m$, $m = j-k$, with Morse indices $m(u_i) = i+k$. In the degenerate case the argument is more complicated and one has to use the structure of $H_G^*(f^a, f^b)$ as a module over $\mathbb{F}_2[w]$. □

4.3 Bifurcation theory

In this section we consider nontrivial solutions of the parameter dependent problem

$$\nabla f_\lambda(u) = 0 \tag{4.2}$$

under the assumption that

$$\nabla f_\lambda(0) = 0$$

holds for all parameters λ.

Definition 4.5 *Let U be an open neighborhood of 0 in the Hilbert space E, let Λ be an open interval and let $f \in \mathcal{C}^1(\Lambda \times U, \mathbb{R})$ be such that $\nabla f_\lambda(0) = 0$ for every $\lambda \in \Lambda$ when $f_\lambda = f(\lambda, \cdot)$. A point $(\lambda_0, 0) \in \Lambda \times U$ is a bifurcation point for equation (4.2) if every neighborhood of $(\lambda_0, 0)$ in $\Lambda \times U$ contains at least one solution (λ, u) of (4.2) such that $u \neq 0$.*

Theorem 4.6 *Let $f \in \mathcal{C}^1(\Lambda \times U, \mathbb{R})$ be as in Definition 4.5. Assume that the following conditions are satisfied :*

i) *0 is an isolated critical point of f_a and f_b for some reals $a < b$ in Λ,*

ii) *for every $a < \lambda < b$, f_λ satisfies the Palais-Smale condition over a closed ball $B[0, R] \subset U$, that is, every (PS)-sequence in $B[0, R]$ has a convergent subsequence.*

iii) *there exists* $n \in \mathbb{N}$ *such that* $C_n(f_a, 0) \not\cong C_n(f_b, 0)$.

Then there exists a bifurcation point $(\lambda_0, 0) \in [a, b] \times \{0\}$ *for equation* (4.2).

Theorem 4.6 is due to Mawhin and Willem (see [39]).

Let us consider a variant of problem (4.1) :

$$
\begin{aligned}
-\Delta u &= \lambda g(u), \quad \text{in } \Omega, \\
u &= 0, \qquad\quad \text{on } \partial\Omega.
\end{aligned}
\tag{4.3}
$$

We assume that g satisfies (g_1) and

(g_6) $g(0) = 0$, $g'(0) = 1$.

The corresponding functional is defined on $H_0^1(\Omega)$ by

$$
f(\lambda, u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega G(u) dx.
$$

Theorem 4.7 *Under assumptions* (g_1) *and* (g_6), $(\lambda, 0)$ *is a bifurcation point for problem* (4.3) *if and only if* λ *is an eigenvalue of* $-\Delta$ *with the Dirichlet boundary condition.*

For the proof we refer to [56].

4.4 Closed geodesics

Let (M, g) be a compact riemannian manifold without boundary. A geodesic is a curve $c : I \to M$, $I \subset \mathbb{R}$ an interval, satisfying the differential equation $\nabla_{\dot{c}}\, \dot{c} = 0$, that is the tangent field $\dot{c}$ is tangent along c. A periodic geodesic $c : \mathbb{R} \to M$ is said to be closed.

Closed geodesics are critical points of the energy functional

$$
f : H^1(S^1, M) \to \mathbb{R}, \quad f(c) = \int_0^1 \|\dot{c}\|^2 dt
$$

where $S^1 = \mathbb{R}/\mathbb{Z}$. They are also critical points of the lenght functional $L(c) = \int_0^1 \|\dot{c}\| dt$. However, this functional is invariant under reparametrizations which implies that given a nonconstant closed geodesic c, all reparametrizations $c \circ \sigma$, $\sigma : \mathbb{R} \to \mathbb{R}$ a strictly increasing C^1-map such that $\sigma(t + 1) = \sigma(t) + 1$, are also critical points of L at the same level $L(c) = L(c \circ \sigma)$. It follows that the Palais-Smale condition cannot hold for L. On the contrary, a critical point of f is automatically parametrized proportional to arc-length. The functional f does satisfy the Palais-Smale condition.

Although being a classical problem of Morse theory, the problem of the existence of closed geodesics has new features not present in our discussion so far. First of all observe that although f is not invariant under reparametrizations, it is invariant under time shifts. Given a 1-periodic H^1-function $c : \mathbb{R} \to M$ and given $\tau \in \mathbb{R}$, we define $c_\tau(t) := c(t+\tau)$. This defines an action of $S^1 = \mathbb{R}/\mathbb{Z}$ on $H^1(S^1, M)$ and clearly f is invariant under this action: $f(c_\tau) = f(c)$. As a consequence, if c is a critical point of f, so is c_τ and therefore f does not have isolated critical points. Instead each critical point corresponding to a nonconstant closed geodesic yields a manifold c_τ, $\tau \in \mathbb{R}/\mathbb{Z}$, of critical points. The local theory developed in Section 3 can be extended to cover manifolds of critical points. This goes back to the work of Bott [11]; see also [39, Chapter 10] for a presentation with applications to differential equations.

The S^1-invariance of f can be used very successfully to obtain "many" critical points. We have seen already in section 4.2 that symmetry implies the existence of multiple critical points. For the closed geodesic problem yet another difficulty appears. If c is a critical point of f then $c^m(t) := c(mt)$ is also a critical point of f, any $m \in \mathbb{N}$. Geometrically, c and c^m, $m > 1$, describe the same closed geodesic and should not be counted separately. A closed geodesic is said to be prime if it is not the m-th iterate, $m > 1$, of a geodesic. In order to understand the contribution of c and its iterates c^m Bott [12] developed an iteration theory for closed geodesics.

A discussion of the Morse theory for the closed geodesics problem goes far beyond the scope of this survey. We state two important results where Morse theory played a decisive role.

Theorem 4.8 *If M is simply connected and if the Betti numbers of the free loop space $H^1(S^1, M) \simeq C^0(S^1, M)$ with respect to some field of coefficients form an unbounded sequence then there exist infinitely many, geometrically different prime closed geodesics on M. The hypothesis on the free loop space is satisfied if the cohomology algebra $H^*(M; \mathbb{Q})$ is not generated (as a $\mathbb{Q}$-algebra with unit) by a single element.*

Since f is bounded below and satisfies the Palais-Smale condition it has infinitely many critical points provided there are infinitely many non-zero Betti numbers. The condition that the Betti numbers form an unbounded sequence can be used to show that the infinitely many critical points are not just the multiples of only finitely prime closed geodesics. The first statement of Theorem 4.8 is due to Gromoll and Meyer [29], the second purely topological result to Vigué-Poirrier and Sullivan [54]. It applies to many manifolds, in particular to products $M = M_1 \times M_2$ of compact simply connected manifolds. It does not apply to spheres, for instance. The reader may consult the paper [20] for further results, references and a discussion of topological features of the problem.

Theorem 4.9 *On the 2-sphere (S^2, g), any metric g, there are always infinitely many, geometrically different prime closed geodesics.*

This theorem is due to Franks [27] and Bangert [6], covering separate cases. The proof of Bangert is via Morse theory, the proof of Franks uses dynamical systems methods. A Morse theory proof of this part can be found in [33].

Similar problems as in the closed geodesic problem appear for the search of periodic solutions of autonomous Hamiltonian systems. In particular one has an S^1-symmetry, critical points are not isolated, and one has the problem that iterates of critical points are also critical points. We refer the reader to the book [38] for a presentation of the Morse index theory and index formulas for iterated curves for periodic solutions of Hamiltonian systems. In additon to the above mentioned difficulties the corresponding functional is strongly indefinite, Morse indices of critical points are infinite. The next section is devoted to Morse theory for strongly indefinite functionals.

5 Strongly indefinite Morse theory

5.1 Cohomology and relative Morse index

Let E be a real Hilbert space and consider the functional

$$\Phi(x) = \frac{1}{2}\|x^+\|^2 - \frac{1}{2}\|x^-\|^2,$$

where $x^\pm \in E^\pm$ and $E = E^+ \oplus E^-$ is an orthogonal decomposition. Then $\nabla\Phi(x) = x^+ - x^-$, the critical set $K = \{0\}$, the Morse index $M^-(\Phi''(0)) = \dim E^-$ and $C_q(\Phi, 0) = \mathbb{F}$ if $q = \dim E^-$ and 0 otherwise. Hence, if $\dim E^- = +\infty$, then $C_*(\Phi, 0) = 0$, and the critical point 0 will not be seen by the Morse theory developed in sections 2 and 3. In what follows we will be concerned with functionals which are of the form $\Phi(x) = \frac{1}{2}\|x^+\|^2 - \frac{1}{2}\|x^-\|^2 - \psi(x)$, where $E = E^+ \oplus E^0 \oplus E^-$, $\dim E^\pm = +\infty$ and $\nabla\psi$ is compact. If $\Phi \in C^2(E, \mathbb{R})$, then it is easy to see that $M^-(\pm\Phi''(x)) = +\infty$ for any $x \in K$, and therefore the Morse theory developed so far becomes useless. It is also easy to see that functionals of the type described here are strongly indefinite in the sense that they are unbounded below and above on any subspace of finite codimension.

In order to establish a Morse theory which is useful for strongly indefinite functionals we first introduce a suitable cohomology theory [36]. Let $(E_n)_{n=1}^\infty$ be a sequence of closed subspaces of E such that $E_n \subset E_{n+1}$ for all n and $\bigcup_{n=1}^\infty E_n$ is dense in E. We shall call (E_n) a *filtration* of E. For a closed set $X \subset E$ we use the shorthand notation

$$X_n := X \cap E_n.$$

To each E_n we assign a nonnegative integer d_n and we write

$$\mathcal{E} := (E_n, d_n)_{n=1}^\infty.$$

Next, if $(\mathcal{G}_n)_{n=1}^{\infty}$ is a sequence of abelian groups, then we define the asymptotic group

$$[(\mathcal{G}_n)_{n=1}^{\infty}] := \prod_{n=1}^{\infty} \mathcal{G}_n \Big/ \bigoplus_{n=1}^{\infty} \mathcal{G}_n,$$

i.e., $[(\mathcal{G}_n)_{n=1}^{\infty}] = \prod_{n=1}^{\infty} \mathcal{G}_n/\sim$, where $(g_n)_{n=1}^{\infty} \sim (g'_n)_{n=1}^{\infty}$ if and only if $g_n = g'_n$ for all n large enough. In what follows we shall use the shorter notation

$$[\mathcal{G}_n] = [(\mathcal{G}_n)_{n=1}^{\infty}] \text{ and } [\mathcal{G}_n] = \mathcal{G} \text{ if } \mathcal{G}_n = \mathcal{G} \text{ for almost all } n.$$

Let (X, A) be a pair of *closed* subsets of E such that $A \subset X$ and denote the Čech cohomology with coefficients in a field $\mathbb{F}$ by H^*. If $\mathcal{E}$ is as above, then for each $q \in \mathbb{Z}$ we define

$$H^q_{\mathcal{E}}(X, A) := [H^{q+d_n}(X_n, A_n)].$$

Observe that $H^*_{\mathcal{E}} = H^*$ if $E_n = E$ and $d_n = 0$ for almost all n, and in general $H^q_{\mathcal{E}}$ need not be 0 for all $q < 0$. As morphisms in the category of closed pairs we take continuous mappings $f : (X, A) \to (Y, B)$ which preserve the filtration, i.e., $f(X_n) \subset E_n$ for almost all n. Such f will be called *admissible.* If $f_n := f|_{X_n}$, then f induces a homomorphism $f^* : H^q_{\mathcal{E}}(Y, B) \to H^q_{\mathcal{E}}(X, A)$ given by $f^* = [f^*_n]$, where $f^*_n : H^{*+d_n}(Y_n, B_n) \to H^{*+d_n}(X_n, A_n)$. Similarly, the coboundary operator $\delta^* : H^*_{\mathcal{E}}(A) \to H^{*+1}_{\mathcal{E}}(X, A)$ is given by $\delta^* = [\delta^*_n]$. We also define admissible homotopies $G : [0, 1] \times (X, A) \to (Y, B)$ by requiring that $G([0, 1] \times X_n) \subset E_n$ for almost all n. It is easy to see from the definitions and the properties of H^* that $H^*_{\mathcal{E}}$ satisfies the usual Eilenberg-Steenrod axioms for cohomology except the dimension axiom. Moreover, since H^* satisfies the strong excision property, so does $H^*_{\mathcal{E}}$. More precisely, this means that if A, B are closed subsets of E, then there is an isomorphism

$$H^*_{\mathcal{E}}(A, A \cap B) \cong H^*_{\mathcal{E}}(A \cup B, B).$$

The need for strong excision was in fact our reason for using the Čech cohomology.

Let $\tilde{L} : E \to E$ be a linear selfadjoint Fredholm operator such that $\tilde{L}(E_n) \subset E_n$ for almost all n. Then $E = E^+(\tilde{L}) \oplus N(\tilde{L}) \oplus E^-(\tilde{L})$, where $N(\tilde{L})$ is the nullspace (of finite dimension) and $E^{\pm}(\tilde{L})$ are the positive and the negative space of $\tilde{L}$. Since

$$\langle \tilde{L}x^+, y^+\rangle - \langle \tilde{L}x^-, y^-\rangle + \langle x^0, y^0\rangle, \quad x^{\pm}, y^{\pm} \in E^{\pm}(\tilde{L}),\ x^0, y^0 \in N(\tilde{L})$$

is an equivalent inner product, we may (and will) assume without loss of generality that $\tilde{L}x = x^+ - x^-$. We also take

$$d_n := M^-(\tilde{L}|_{E_n}) + d_0 \equiv \dim E^-(\tilde{L})_n + d_0,$$

where $E^-(\tilde{L})_n = E^-(\tilde{L}) \cap E_n$ and d_0 is a convenient normalization constant to be chosen. Let $B : E \to E$ be linear, compact and selfadjoint, $L := \tilde{L} - B$, denote the orthogonal projector on E_n by P_n and the orthogonal projector from the range $R(L)$ of L on $R(L)_n$ by Q_n, and let

$$M^-_{\mathcal{E}}(L) := \lim_{n\to\infty} \left(M^-(Q_nL|_{R(L)_n}) - d_n\right). \tag{5.1}$$

Then $M^-_{\mathcal{E}}(L)$ is a well-defined (and not necessarily positive) integer [36, Proposition 5.2]. Note that $M^-(Q_nL|_{R(L)_n})$ is the Morse index of the quadratic form $\langle Lx, x\rangle$ restricted to $R(L)_n$. It is not too difficult to show that if $N(L) \subset E_{n_0}$ for some n_0, then

$$M^-_{\mathcal{E}}(L) = \lim_{n\to\infty} \left(M^-(P_nL|_{E_n}) - d_n\right) = \lim_{n\to\infty} \Big(M^-(P_nL|_{E_n}) - M^-(\tilde{L}|_{E_n})\Big) - d_0 \tag{5.2}$$

(see [36, Remark 5.1]). Hence $M^-_{\mathcal{E}}(L)$ is a relative Morse index in the sense that it measures the difference between the Morse indices of the operators L and $\tilde{L}$ restricted to E_n (n large). If $N(L) \not\subset E_n$ for any n, then $M^-_{\mathcal{E}}(L)$ may not be equal to the limit in (5.2). A justification why (5.1) and not (5.2) is used as the definition of $M^-_{\mathcal{E}}(L)$ may be found in [36].

Let

$$D := \{x \in E^-(L) : \|x\| \le 1\}, \quad S := \{x \in E^-(L) : \|x\| = 1\}.$$

The connection between $H^*_{\mathcal{E}}$ and $M^-_{\mathcal{E}}(L)$ is expressed in the following

Proposition 5.1 *Suppose L satisfies the conditions formulated above, $N(L) \subset E_n$ and $L(E_n) \subset E_n$ for almost all n. Then $H^q_{\mathcal{E}}(D, S) = \mathbb{F}$ if $q = M^-_{\mathcal{E}}(L)$ and $H^q_{\mathcal{E}}(D, S) = 0$ otherwise.*

Proof. Since $P_nL = LP_n$, the negative space of $L|_{E_n}$ is $E^-(L)_n$. Hence $D_n = \{x \in E^-(L)_n : \|x\| \le 1\}$, so $\dim D_n = M^-_{\mathcal{E}}(L) + d_n$ for almost all n. It follows that $H^{q+d_n}(D_n, S_n) = \mathbb{F}$ if $q = M^-_{\mathcal{E}}(L)$ and 0 otherwise. □

Note in particular that if $\dim E^-(L) = +\infty$, then $H^*(D, S)$ is trivial while $H^*_{\mathcal{E}}(D, S)$ is not.

Remark 5.2 Different though related to $H^*_{\mathcal{E}}$ cohomology theories may be found in [1, 49]. Both theories are constructed by a suitable modification of an infinite-dimensional cohomology of Gęba and Granas [28]. In [3] the reader may find an infinite-dimensional *homology* theory constructed in a very different way, with the aid of a Morse-Witten complex.

5.2 Critical groups and Morse inequalities

The critical groups of an isolated critical point x were defined in section 2.2 by setting $C_q(\Phi, x) = H_q(\Phi^c, \Phi^c \setminus \{x\})$. It may seem natural to define $C_{\mathcal{E}}^q$ here by simply replacing H_q with $H_{\mathcal{E}}^q$. However, this is not possible because $H_{\mathcal{E}}^q(X, A)$ has been defined for closed sets only, and moreover, if x is an isolated critical point of Φ, it is not clear in general how the critical set for $\Phi|_{E_n}$ looks like in small neighborhoods of x, not even if $x \in E_{n_0}$ for some n_0.

We shall construct a Morse theory for functionals which are of the form

$$\Phi(x) = \frac{1}{2}\langle \tilde{L}x, x\rangle - \psi(x) \equiv \frac{1}{2}\|x^+\|^2 - \frac{1}{2}\|x^-\|^2 - \psi(x), \tag{5.3}$$

where $x = x^+ + x^0 + x^- \in E = E^+ \oplus E^0 \oplus E^-$, $\dim E^0 < \infty$, and $\nabla\psi$ is a compact operator. Although it is possible to allow a larger class of Φ (see [36]), there can be no useful Morse theory which includes all smooth Φ such that $M^-(\pm\Phi''(x)) = +\infty$ whenever $x \in K$. This is a consequence of a result by Abbondandolo and Majer [4].

In order to avoid the problem with the critical set of $\Phi|_{E_n}$ mentioned above we shall use an approach which goes back to Gromoll and Meyer (see e.g. [17]). Its main feature is that to each isolated critical point x one assigns a pair (W, W^-) of closed sets such that x is in the interior of W, W^- is the exit set (from W) for the flow of $-\nabla\Phi$, and then one defines the critical groups of x by setting $C_*(\Phi, x) = H_*(W, W^-)$ (in homology theory) and $C^*(\Phi, x) = H^*(W, W^-)$ (in cohomology theory). Since $\nabla\Phi$ is not admissible for $H_{\mathcal{E}}^*$, we shall need a class of mappings which are related to $\nabla\Phi$ and admissible. The results we summarize below may be found, in a more general form, in [36].

Let E be a real Hilbert space, (E_n) a filtration and suppose $\Phi \in C^1(E, \mathbb{R})$. A sequence (x_j) is said to be a $(PS)^*$*-sequence* (with respect to (E_n)) if $\Phi(x_j)$ is bounded, $x_j \in E_{n_j}$ for some n_j, $n_j \to \infty$ and $P_{n_j}\nabla\Phi(x_j) \to 0$ as $j \to \infty$. If each $(PS)^*$-sequence has a convergent subsequence, then Φ is said to satisfy the $(PS)^*$*-condition*. Using the density of $\bigcup_{n=1}^\infty E_n$ it is easy to show that convergent subsequences of (x_j) tend to critical points and $(PS)^*$ implies the Palais-Smale condition.

Lemma 5.3 *If Φ is of the form (5.3), then each bounded $(PS)^*$-sequence has a convergent subsequence.*

Proof. Let (x_j) be a bounded $(PS)^*$-sequence. Then $x_j \rightharpoonup \bar{x}$ and $\nabla\psi(x_j) \to w$ for some $w \in E$ after passing to a subsequence. If $x_j = y_j + x_j^0$, $y_j \in R(\tilde{L})$, $x_j^0 \in E^0 \equiv N(\tilde{L})$, then $\tilde{L}y_j - P_{n_j}\nabla\psi(x_j) \to 0$, and since E^0 is finite-dimensional and $\tilde{L}|_{R(\tilde{L})}$ invertible, $x_j \to \bar{x}$ after passing to a subsequence once more. □

In order to construct flows which are admissible mappings for the cohomology theory $H_{\mathcal{E}}^*$ we need to modify the notion of pseudogradient. Let $Y \subset E \setminus K$. A mapping

$V : Y \to E$ is said to be a *gradient-like* vector field for Φ on Y if V is locally Lipschitz continuous, $\|V(x)\| \le 1$ for all $x \in Y$ and there is a function $\beta : Y \to \mathbb{R}^+$ such that $\langle \nabla\Phi(x), V(x)\rangle \ge \beta(x)$ for all $x \in Y$ and $\inf_{x\in Z} \beta(x) > 0$ for any set $Z \subset Y$ which is bounded away from K and such that $\sup_{x\in Z} |\Phi(x)| < \infty$.

Lemma 5.4 ([36], Lemma 2.2) *If U is an open subset of E and Φ satisfies (5.3) and $(PS)^*$, then there exists a gradient-like and filtration-preserving vector field V on $U \setminus K$.*

Suppose now A is an isolated compact subset of K. A pair (W, W^-) of closed subsets of E is said to be an *admissible pair* for Φ and A with respect to $\mathcal{E}$ if: (i) W is bounded away from $K \setminus A$, $W^- \subset \mathrm{bd}(W)$ and $A \subset \mathrm{int}(W)$ (bd and int respectively denote the boundary and the interior), (ii) $\Phi|_W$ is bounded, (iii) there exist a neighborhood N of W and a filtration-preserving gradient-like vector field V for Φ on $N \setminus A$, (iv) W^- is the union of finitely many (possibly intersecting) closed sets each lying on a C^1-manifold of codimension 1, $V|_{W^-}$ is transversal to these manifolds, the flow φ of $-V$ can leave W only through W^- and if $x \in W^-$, then $\varphi^t(x) \notin W$ for any $t > 0$.

Since W is bounded away from $K \setminus A$, it is easy to see that for each neighborhood $U \subset W$ of A the critical points of $\Phi|_{W_n}$ are contained in U_n provided n is large enough. The following two results are basic for our Gromoll-Meyer type approach to Morse theory:

Proposition 5.5 ([36], Proposition 2.5) *Suppose Φ satisfies (5.3), $(PS)^*$ and $\Phi(K) \subset (a, b)$. Then (Φ_a^b, Φ_a^a) is an admissible pair for Φ and K.*

Proposition 5.6 ([36], Propositions 2.6 and 2.7) *Suppose Φ satisfies (5.3), $(PS)^*$ and p is an isolated critical point. Then for each open neighborhood U of p there exists an admissible pair (W, W^-) for Φ and p such that $W \subset U$. Moreover, if $(\tilde{W}, \tilde{W}^-)$ is another admissible pair, then $H^*_{\mathcal{E}}(W, W^-) \cong H^*_{\mathcal{E}}(\tilde{W}, \tilde{W}^-)$.*

The existence part of this proposition is shown by considering the flow defined by

$$\frac{d\varphi}{dt} = -\chi(\varphi)V(\varphi), \quad \varphi(0, x) = x,$$

where χ is a cutoff function at $x = p$. Choosing a small $\varepsilon > 0$ and a sufficiently small ball $B_\delta(p) \subset \Phi_{c-\varepsilon}$, where $c = \Phi(p)$, we can take

$$W = \{\varphi^t(x) : t \ge 0,\ x \in \bar{B}_\delta(p),\ \varphi^t(x) \in \Phi_{c-\varepsilon}\} \text{ and } W^- = W \cap \Phi_{c-\varepsilon}^{c-\varepsilon}.$$

The second part of the proposition is also shown by cutting off the flow of V in a suitable way. The proof is rather technical and the strong excision property of $H^*_{\mathcal{E}}$ comes to an essential use here.

If x is an isolated critical point and (W, W^-) an admissible pair for Φ and x, we set

$$C^q_{\mathcal{E}}(\Phi, x) := H^q_{\mathcal{E}}(W, W^-), \quad q \in \mathbb{Z}.$$

According to Proposition 5.6 the critical groups $C^q_{\mathcal{E}}(\Phi, x)$ are well defined by the above formula. If the set K_c is finite and isolated in K for some c, then for each $x_i \in K_c$ there exists an admissible pair (W_i, W_i^-), and we may assume $W_i \cap W_j = \emptyset$ whenever $i \neq j$. So in an obvious notation and by the argument of Lemma 2.5,

$$C^q_{\mathcal{E}}(\Phi, K_c) = \bigoplus_{x \in K_c} C^q_{\mathcal{E}}(\Phi, x), \quad q \in \mathbb{Z}.$$

Let

$$\dim_{\mathcal{E}} H^q_{\mathcal{E}}(X, A) := [\dim H^{q+d_n}(X, A)] \in [\mathbb{Z}] = \prod_{n=1}^{\infty} \mathbb{Z} \Big/ \bigoplus_{n=1}^{\infty} \mathbb{Z}$$

(to be more precise, $[\dim H^{q+d_n}(X, A)] \in [\mathbb{N}_0]$ because all the dimensions are of course nonnegative). If $\dim H^{q+d_n}(X, A) = d$ for almost all n, according to our earlier convention we write $\dim_{\mathcal{E}} H^q_{\mathcal{E}}(X, A) = d$. Suppose K is finite and $\Phi(K) \subset (a, b)$. As in section 2.2, we may define

$$m^q_{\mathcal{E}} := \sum_j \dim_{\mathcal{E}} C^q_{\mathcal{E}}(\Phi, K_{c_j}), \qquad p^q_{\mathcal{E}} := \dim_{\mathcal{E}} H^q_{\mathcal{E}}(\Phi^b_a, \Phi^a_a)$$

and

$$M^{\Phi}_{\mathcal{E}}(t; a, b) := \sum_{q \in \mathbb{Z}} m^q_{\mathcal{E}} t^q, \qquad P^{\Phi}_{\mathcal{E}}(t; a, b) := \sum_{q \in \mathbb{Z}} p^q_{\mathcal{E}} t^q$$

(note that (Φ^b_a, Φ^a_a) is an admissible pair for Φ and K according to Proposition 5.5). These are formal Laurent series with coefficients in $[\mathbb{N}_0]$. If $m^q_{\mathcal{E}}$ and $p^q_{\mathcal{E}}$ are 0 for all but finitely many q, then $M^{\Phi}_{\mathcal{E}}(t; a, b), P^{\Phi}_{\mathcal{E}}(t; a, b) \in [\mathbb{N}_0]\,[t, t^{-1}]$, where $[\mathbb{N}_0][t, t^{-1}]$ is the set of Laurent polynomials with coefficients in $[\mathbb{N}_0]$. If the coefficients are the same for almost all n, we write $M^{\Phi}_{\mathcal{E}}(t; a, b), P^{\Phi}_{\mathcal{E}}(t; a, b) \in \mathbb{N}_0[t, t^{-1}]$.

Theorem 5.7 (Morse inequalities, [36], Theorem 3.1 and Corollary 3.3) *Suppose Φ satisfies (5.3), $(PS)^*$, K is finite, $\Phi(K) \subset (a, b)$ and $m^q_{\mathcal{E}} = 0$ for almost all $q \in \mathbb{Z}$. Then $p^q_{\mathcal{E}} = 0$ for almost all $q \in \mathbb{Z}$ and there exists $Q_{\mathcal{E}} \in [\mathbb{N}_0[t, t^{-1}]]$ such that*

$$M^{\Phi}_{\mathcal{E}}(t; a, b) = P^{\Phi}_{\mathcal{E}}(t; a, b) + (1 + t) Q_{\mathcal{E}}(t).$$

The proof is rather similar to that of Theorem 2.6; however, in lack of (A2) more complicated sets than Φ^{c_j} and $\Phi^{c_j} \setminus K_{c_j}$ need to be used.

In order to apply the Morse inequalities we need to be able to perform local computations.

Theorem 5.8 ([36], Theorem 5.3) *Suppose Φ satisfies (5.3), $(PS)^*$, p is an isolated critical point of Φ and*

$$\Phi(x) = \Phi(p) + \frac{1}{2}\langle L(x-p), x-p\rangle - \tilde{\psi}(x), \tag{5.4}$$

where L is invertible and $\nabla\tilde{\psi}(x) = o(\|x-p\|)$ as $x \to p$. Then $C^q_{\mathcal{E}}(\Phi, p) = \mathbb{F}$ for $q = M^-_{\mathcal{E}}(L)$ and 0 otherwise.

We note that $M^-_{\mathcal{E}}(L)$ is well defined and finite because $L - \tilde{L}$ is compact according to (5.3).

Suppose now $\Phi \in C^2(U, \mathbb{R})$, where U is a neighborhood of an isolated critical point p. Then (5.4) holds, $\dim N(L) < \infty$, $\nabla\Phi(p) = 0$, $\Phi''(p) = L$ and

$$\nabla\Phi(p+z+y) = Ly - \nabla\tilde{\psi}(p+z+y),$$

where $x = p+z+y$, $z \in N(L)$, $y \in R(L)$. Denote the orthogonal projector on $R(L)$ by Q. Since $L|_{R(L)}$ is invertible, we may use the implicit function theorem in order to obtain $\delta > 0$ and a C^1-mapping $\alpha : B_\delta(0) \cap N(L) \to R(L)$ such that $\alpha(0) = 0$, $\alpha'(0) = 0$ and

$$Q\nabla\Phi(p+z+\alpha(z)) \equiv 0.$$

Letting

$$g(z) := \Phi(p+z+\alpha(z)) - \Phi(p) = \frac{1}{2}\langle L\alpha(z), \alpha(z)\rangle - \tilde{\psi}(p+z+\alpha(z)),$$

one readily verifies that 0 is an isolated critical point of g, hence the critical groups $C^q(g, 0)$ of section 2.2 are well defined (however, we use cohomology instead of homology here).

Theorem 5.9 (Shifting theorem, [36], Theorem 5.4) *Suppose Φ satisfies (5.3), $(PS)^*$ and $\Phi \in C^2(U, \mathbb{R})$, where U is a neighborhood of an isolated critical point p. Then*

$$C^q_{\mathcal{E}}(\Phi, p) = C^{q - M^-_{\mathcal{E}}(L)}(g, 0), \quad q \in \mathbb{Z}.$$

Remark 5.10 Other Morse theories for strongly indefinite functionals may be found e.g. in the papers [1, 3, 49] already mentioned and also in [2, 31, 32]. With an exception of [3] they are similar in essence but differ by the way the technical issues have been resolved and by the range of applications. Each of them also has certain advantages and disadvantages. We would also like to mention that we have neither touched upon equivariant Morse theory for strongly indefinite functionals [35] nor upon the Floer and Floer-Conley theories (see e.g. [5], [40], [46]).

6 Strongly indefinite variational problems

6.1 Hamiltonian systems

As an application of Morse theory for strongly indefinite functionals we shall consider the problem of existence of periodic solutions to Hamiltonian systems

$$\dot z = JH_z(z,t), \qquad z \in \mathbb{R}^{2N}, \tag{6.1}$$

where

$$J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

is the standard symplectic matrix. We shall need the following assumptions on H:

(H_1) $H \in C(\mathbb{R}^{2N} \times \mathbb{R}, \mathbb{R})$, $H_z \in C(\mathbb{R}^{2N} \times \mathbb{R}, \mathbb{R}^{2N})$ and $H(0,t) \equiv 0$;

(H_2) H is 2π-periodic in the t-variable;

(H_3) $|H_z(z,t)| \le c(1+|z|^{s-1})$ for some $c>0$ and $s \in (2,\infty)$;

(H_4) $H_{zz} \in C(\mathbb{R}^{2N} \times \mathbb{R}, \mathbb{R}^{4N^2})$;

(H_5) $|H_{zz}(z,t)| \le d(1+|z|^{s-2})$ for some $d>0$ and $s \in (2,\infty)$.

When (H_2) is satisfied, the natural period for solutions of (6.1) is 2π. It is clear that the assumption $H(0,t) \equiv 0$ in (H_1) causes no loss of generality and if (H_5) is assumed, then (H_3) necessarily holds. We also remark that any period T in (H_2) may be normalized to 2π by a simple change of the t-variable.

Below we give a short account of a variational setup for periodic solutions of (6.1). We follow [8] where more details and references may be found. Let $E := H^{1/2}(S^1, \mathbb{R}^{2N})$ be the Sobolev space of 2π-periodic $\mathbb{R}^{2N}$-valued functions

$$z(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos kt + b_k \sin kt, \qquad a_0, a_k, b_k \in \mathbb{R}^{2N} \tag{6.2}$$

such that $\sum_{k=1}^{\infty} k(|a_k|^2 + |b_k|^2) < \infty$. Then E is a Hilbert space with an inner product

$$\langle z, w\rangle := 2\pi a_0 \cdot a_0' + \pi \sum_{k=1}^{\infty} k(a_k \cdot a_k' + b_k \cdot b_k')$$

(a_k', b_k' are the Fourier coefficients of w). It is well known that the Sobolev embedding $E \hookrightarrow L^q(S^1, \mathbb{R}^{2N})$ is compact for any $q \in [1,\infty)$ but $E \not\subset L^\infty(S^1, \mathbb{R}^{2N})$. Let

$$\Phi(z) := \frac{1}{2}\int_0^{2\pi} (-J\dot z \cdot z)\, dt - \int_0^{2\pi} H(z,t)\, dt \equiv \langle \tilde L z, z\rangle - \psi(z). \tag{6.3}$$

Proposition 6.1 ([8], Proposition 2.1) *If H satisfies (H_{1-2-3}), then $\Phi \in C^1(E,\mathbb{R})$ and $\nabla\Phi(z) = 0$ if and only if z is a 2π-periodic solution of (6.1). Moreover, $\nabla\psi$ is completely continuous in the sense that $\nabla\psi(z_j) \to \nabla\psi(z)$ whenever $z_j \rightharpoonup z$. If, in addition, H satisfies (H_4) and (H_5), then $\Phi \in C^2(E,\mathbb{R})$ and $\psi''(z)$ is a compact linear operator for all $z \in E$.*

Suppose $z(t) = a_k \cos kt \pm Ja_k \sin kt$. Then

$$\langle \tilde{L}z, z\rangle = \int_0^{2\pi} (-J\dot{z}\cdot z)\,dt = \pm 2\pi k|a_k|^2 = \pm\|z\|^2 \tag{6.4}$$

by a simple computation and it follows that E has the orthogonal decomposition $E = E^+ \oplus E^0 \oplus E^-$, where

$$E^0 = \{z \in E : z = a_0 \in \mathbb{R}^{2N}\},$$
$$E^\pm = \left\{ z \in E : z(t) = \sum_{k=1}^{\infty} a_k \cos kt \pm Ja_k \sin kt,\ a_k \in \mathbb{R}^{2N} \right\}.$$

According to (6.4),

$$\langle \tilde{L}z, z\rangle = \|z^+\|^2 - \|z^-\|^2 \qquad (z = z^+ + z^0 + z^- \in E^+ \oplus E^0 \oplus E^-),$$

and since $E^\pm$ are infinite-dimensional, Φ is strongly indefinite. Let

$$E_n := \left\{ z \in E : z(t) = a_0 + \sum_{k=1}^{n} a_k \cos kt + b_k \sin kt \right\},$$

then (E_n) is a filtration of E and $\tilde{L}(E_n) \subset E_n$ for all n. Set

$$d_n := N(1+2n) \equiv M^-(\tilde{L}|_{E_n}) + N \tag{6.5}$$

(hence $d_0 = N$ in the notation of section 5.1).

Suppose A is a symmetric $2N \times 2N$ constant matrix and let

$$\langle Bz, w\rangle := \int_0^{2\pi} Az\cdot w\,dt. \tag{6.6}$$

Then B is a selfadjoint operator on E and it follows from the compactness of the embedding $E \hookrightarrow L^2(S^1,\mathbb{R}^{2N})$ that B is compact. A simple computation using (6.2) shows that setting $L := \tilde{L} - B$, we have

$$\langle Lz, z\rangle = -2\pi Aa_0 \cdot a_0 + \pi \sum_{k=1}^{\infty} k\left((-Jb_k - \frac{1}{k}Aa_k)\cdot a_k + (Ja_k - \frac{1}{k}Ab_k)\cdot b_k \right). \tag{6.7}$$

Note in particular that $L(E_n) \subset E_n$ for all n. The restriction of the quadratic form $\langle Lz, z\rangle$ to a subspace corresponding to a fixed $k \geq 1$ is represented by the $4N \times 4N$ matrix $\pi k T_k(A)$, where

$$T_k(A) := \begin{pmatrix} -\frac{1}{k}A & -J \\ J & -\frac{1}{k}A \end{pmatrix}.$$

For a symmetric matrix C, set $M^+(C) := M^-(-C)$ and let $M^0(C)$ be the nullity of C. The matrix

$$\begin{pmatrix} 0 & -J \\ J & 0 \end{pmatrix}$$

has the eigenvalues ± 1, both of multiplicity $2N$, hence $M^{\pm}(T_k(A)) = 2N$ for all k large enough. Therefore

$$\begin{aligned} i^-(A) &:= M^+(A) - N + \sum_{k=1}^{\infty}(M^-(T_k(A)) - 2N), \\ i^+(A) &:= M^-(A) - N + \sum_{k=1}^{\infty}(M^+(T_k(A)) - 2N), \\ i^0(A) &:= M^0(A) + \sum_{k=1}^{\infty} M^0(T_k(A)) \end{aligned}$$

are well defined finite numbers and it is not difficult to see that $i^-(A)+i^0(A)+i^+(A) = 0$. Again, we refer to [8] for more details and references. It follows using (6.7) that $i^0(A) = \dim N(L)$ is the number of linearly independent 2π-periodic solutions of the linear system $\dot{z} = JAz$ (so in particular, $\dim N(L) \leq 2N$) and $N(L) \subset E_n$ for almost all n. It can be further seen that L is invertible if and only if $\sigma(JA) \cap i\mathbb{Z} = \emptyset$, where σ denotes the spectrum (this is called the nonresonance condition because the linear system above has $z = 0$ as the only 2π-periodic solution). Also, using (6.7) again,

$$\dim E_n^-(L) = M^+(A) + \sum_{k=1}^{n} M^-(T_k(A)),$$

hence by (5.2) and (6.5), $M^-_{\mathcal{E}}(L) = i^-(A)$. Similarly, $M^+_{\mathcal{E}}(L) := M^-_{\mathcal{E}}(-L) = i^+(A)$. We have sketched a proof of the following

Proposition 6.2 ([36], Proposition 7.1) $\dim N(L) = i^0(A)$ *and* $M^{\pm}_{\mathcal{E}}(L) = i^{\pm}(A)$.

Remark 6.3 The number $i^-(A)$ (and thus $M^-_{\mathcal{E}}(L)$) equals the Maslov index of the fundamental solution of the system $\dot{z} = JAz$. For comments and references, see [8, Remark 2.8] and [36, Remark 7.2].

Assume that H_z satisfies the following asymptotic linearity conditions at 0 and infinity:
(6.8)

$$H(z,t) = \frac{1}{2}A_\infty z \cdot z + G_\infty(z,t), \text{ where } (G_\infty)_z(z,t) = o(|z|) \text{ uniformly in } t \text{ as } |z| \to \infty$$

and

$$(6.9) \quad H(z,t) = \frac{1}{2}A_0 z \cdot z + G_0(z,t), \text{ where } (G_0)_z(z,t) = o(z) \text{ uniformly in } t \text{ as } z \to 0.$$

Here A_∞ and A_0 are constant symmetric $2N \times 2N$ matrices. It is clear that (6.8) implies (H_3) for any $s > 2$. We shall use the notation $L_\infty = \tilde{L} - B_\infty$ and $L_0 = \tilde{L} - B_0$, where B_∞, B_0 are the operators defined in (6.6), with A replaced by A_∞ and A_0 respectively. We also set

$$\psi_\infty(z) := \int_0^{2\pi} G_\infty(z,t)\,dt \quad \text{and} \quad \psi_0(z) := \int_0^{2\pi} G_0(z,t)\,dt.$$

It is easy to show [8, Lemma 2.4] that $\nabla\psi_\infty(z) = o(\|z\|)$ as $\|z\| \to \infty$ and $\nabla\psi_0(z) = o(\|z\|)$ as $z \to 0$.

Lemma 6.4 *Suppose H satisfies (H_{1-2}) and (6.8). If $\sigma(JA_\infty) \cap i\mathbb{Z} = \emptyset$, then the functional Φ satisfies the $(PS)^*$-condition.*

Proof. Let (z_j) be a $(PS)^*$-sequence. Then

$$P_{n_j}\nabla\Phi(z_j) = L_\infty z_j - P_{n_j}\nabla\psi_\infty(z_j) \to 0,$$

so (z_j) is bounded because L_∞ is invertible and $\nabla\psi_\infty(z_j)/\|z_j\| \to 0$ if $\|z_j\| \to \infty$. Now it remains to invoke Lemma 5.3 (with $\tilde{L}$ and ψ given by (6.3)). □

A 2π-periodic solution z_0 of (6.1) is said to be *nondegenerate* if $w = 0$ is the only 2π-periodic solution of the system $\dot{w} = JH_{zz}(z_0(t))w$. It is easy to see that z_0 is nondegenerate if and only if $\Phi''(z_0)$ is invertible.

Theorem 6.5 ([36], Theorem 7.4 and Remark 7.7) *Suppose H satisfies (H_{1-2}), (6.8), (6.9) and $\sigma(JA_\infty) \cap i\mathbb{Z} = \sigma(JA_0) \cap i\mathbb{Z} = \emptyset$. If $i^-(A_\infty) \neq i^-(A_0)$, then (6.1) has a nontrivial 2π-periodic solution z_0. Moreover, if H satisfies (H_{4-5}) and z_0 is nondegenerate, then (6.1) has a second nontrivial 2π-periodic solution.*

Proof. It follows from our earlier considerations and from the hypotheses that $L_\infty(E_n) \subset E_n$, $L_0(E_n) \subset E_n$, $N(L_\infty) = N(L_0) = \{0\}$, $M^-_{\mathcal{E}}(L_\infty) = i^-(A_\infty)$ and $M^-_{\mathcal{E}}(L_0) = i^-(A_0)$.

Suppose 0 is the only 2π-periodic solution of (6.1). Consider the functional $I(z) := \frac{1}{2}\langle L_\infty z, z\rangle$ whose only critical point is 0, and for $R_0 > 0$ let

$$(6.10)\qquad \begin{aligned} W &:= \{z = w^+ + w^- \in E^+(L_\infty) \oplus E^-(L_\infty) : \langle \pm L_\infty w^\pm, w^\pm\rangle \le R_0\} \\ W^- &:= \{z \in W : \langle L_\infty w^-, w^-\rangle = -R_0\}. \end{aligned}$$

It is easy to see that the mapping $V_1(z) = \|L_\infty z\|^{-1} L_\infty z$ is a gradient-like vector field for I on $E \setminus \{0\}$, it preserves the filtration and (W, W^-) is an admissible pair for I and 0. Hence by Theorem 5.8,

$$(6.11)\qquad H^q_{\mathcal{E}}(W, W^-) = \delta^q_{i^-(A_\infty)}\mathbb{F}.$$

Since $\nabla\psi_\infty(z) = o(\|z\|)$ as $\|z\| \to \infty$, V_1 is also gradient-like for Φ on $E \setminus B_R(0)$ provided R is large enough. By Lemma 5.4 there exists a gradient-like and filtration-preserving vector field V_2 on $B_{R+1}(0) \setminus \{0\}$. Setting $V = \chi_1 V_1 + \chi_2 V_2$, where $\{\chi_1, \chi_2\}$ is a partition of unity subordinated to the cover $\{E \setminus \bar{B}_R(0), B_{R+1}(0) \setminus \{0\}\}$ of $E \setminus \{0\}$, one verifies using this V that (W, W^-) is an admissible pair for Φ and 0 if R_0 is sufficiently large. Therefore by Proposition 5.6, $H^q_{\mathcal{E}}(W, W^-) = C^q_{\mathcal{E}}(\Phi, 0) = \delta^q_{i^-(A_0)}\mathbb{F}$, a contradiction to (6.11).

It is clear that (W, W^-) is also an admissible pair for Φ and $\{0, z_0\}$, possibly after taking larger R and R_0. If z_0 is nondegenerate, then $C^{q_0}_{\mathcal{E}}(\Phi, z_0) = \delta^q_{q_0}\mathbb{F}$ for some $q_0 \in \mathbb{Z}$, hence choosing $t = -1$ in Theorem 5.7 we obtain

$$(-1)^{i^-(A_0)} + (-1)^{q_0} = (-1)^{i^-(A_\infty)},$$

a contradiction again. □

It is in fact not necessary to assume (H_5). It has been shown in [36] that (H_4) implies the existence (but not continuity) of $\Phi''(z_0)$. We also remark that the existence of one nontrivial solution can be shown without using Morse theory, with the aid of a linking argument [8]. And since it is in general not possible to verify whether the solution z_0 is nondegenerate, we can only make a heuristic statement that "usually" this will be the case. Below we give a sufficient condition for (6.1) to have two nontrivial solutions regardless of any nondegeneracy assumption.

Theorem 6.6 ([10], Theorem 2.3, [36], Theorem 7.8) *Suppose H satisfies $(H_{1-2-4-5})$, (6.8), (6.9) and $\sigma(JA_\infty) \cap i\mathbb{Z} = \sigma(JA_0) \cap i\mathbb{Z} = \emptyset$. If $|i^-(A_\infty) - i^-(A_0)| \ge 2N$, then (6.1) has at least 2 nontrivial 2π-periodic solutions.*

Proof. Let z_0 be the nontrivial solution obtained in the preceding theorem and suppose there are no other ones. By Theorem 5.9,

$$C^q_{\mathcal{E}}(\Phi, z_0) = C^{q-r_0}(g, 0),$$

where $r_0 = M_{\mathcal{E}}^-(\Phi''(z_0))$ and g is defined in an open neighborhood of 0 in $N(\Phi''(z_0))$. Since $\dim N(\Phi''(z_0)) \le 2N$, $C_{\mathcal{E}}^q(\Phi, z_0) = 0$ for $q - r_0 < 0$ and $q - r_0 > 2N$. Moreover, if g has a local minimum at 0, then $C^{q-r_0}(g, 0) \neq 0$ if and only if $q - r_0 = 0$, and if g has a local maximum there, $C^{q-r_0}(g, 0) \neq 0$ if and only if $q - r_0 = \dim N(\Phi''(z_0))$. Otherwise $C^0(g, 0) = C^{2N}(g, 0) = 0$ (cf. Corollary 3.8). Therefore there exists $q_0 \in \mathbb{Z}$ such that $C_{\mathcal{E}}^q(\Phi, z_0) = 0$ whenever $q < q_0$ and $q > q_0 + 2N - 2$. Hence by (6.11) and the Morse inequalities,

$$t^{i^-(A_0)} + \sum_{q=q_0}^{q_0+2N-2} b_q t^q = t^{i^-(A_\infty)} + (1+t)Q_{\mathcal{E}}(t),$$

where $b_q \in [\mathbb{N}_0]$. Since there is an exponent $i^-(A_\infty)$ on the right-hand side, we must have $q_0 \le i^-(A_\infty) \le q_0 + 2N - 2$. Moreover, $q_0 - 1 \le i^-(A_0) \le q_0 + 2N - 1$. To see this, suppose $i^-(A_0) \le q_0 - 2$ (the other case is similar). Then on the left-hand side there is an exponent $i^-(A_0)$ but no exponents $i^-(A_0) \pm 1$ which is impossible for the right-hand side. Now combining the inequalities above we obtain $|i^-(A_\infty) - i^-(A_0)| \le 2N - 1$, a contradiction. □

As our final application we consider the system (6.1) with H being 2π-periodic in all variables. It is clear that if z_0 is a 2π-periodic solution of (6.1), so is $z_k(t) = z_0(t) + 2\pi k$, $k \in \mathbb{Z}^{2N}$. We shall call two solutions z_1, z_2 *geometrically distinct* if $z_1 - z_2 \not\equiv 2\pi k$ for any $k \in \mathbb{Z}^{2N}$. Let $z = x + v$, $x \in \tilde{E} := E^+ \oplus E^-$, $v \in E^0$. Since $N(\tilde{L}) = E^0 \equiv \mathbb{R}^{2N}$, we may redefine Φ by setting

$$\Phi(x, v) = \frac{1}{2}\int_0^{2\pi} (-J\dot{x} \cdot x)\, dt - \int_0^{2\pi} H(x + v, t)\, dt = \frac{1}{2}\langle \tilde{L}x, x\rangle - \psi(x, v).$$

The periodicity of H with respect to $z_1, \ldots, z_{2N}$ implies $\Phi(x, v_1) = \Phi(x, v_2)$ whenever $v_1 \equiv v_2 \pmod{2\pi}$. Therefore v may be regarded as an element of the torus $T^{2N} = \mathbb{R}^{2N}/2\pi\mathbb{Z}^{2N}$ and $\Phi \in C^1(M, \mathbb{R})$, where $M := \tilde{E} \times T^{2N}$. The advantage of such representation of Φ is that distinct critical points of Φ on M correspond to geometrically distinct solutions of (6.1).

Theorem 6.7 *Suppose H is 2π-periodic in all variables and satisfies (H_{1-4}). If all 2π-periodic solutions of (6.1) are nondegenerate, then the number of geometrically distinct ones is at least 2^{2N}.*

This is the second part of the celebrated result by Conley and Zehnder on Arnold's conjecture [21]. The first part asserts that without the nondegeneracy assumption the number of geometrically distinct 2π-periodic solutions is at least $2N + 1$ (see [8, section 2.6] for a sketch of a proof).

Proof. We outline the argument. Since the periodicity implies (H_5), $\Phi \in C^2(M,\mathbb{R})$. Let $\mathcal{E} := (M_n, d_n)$, where $M_n = \tilde{E}_n \times T^{2N}$ and $d_n = 2nN$. According to [36, Remark 2.15], the theory developed in section 5.2 still applies. Suppose Φ has only finitely many critical points and define $\tilde{I}(x) = \frac{1}{2}\langle \tilde{L}x, x\rangle$. Then $\tilde{I} : \tilde{E} \to \tilde{E}$, 0 is the only critical point of $\tilde{I}$ and $(\tilde{W}, \tilde{W}^-)$ is an admissible pair for $\tilde{I}$ and 0, where $(\tilde{W}, \tilde{W}^-)$ is defined in the same way as (W, W^-) in (6.10), but with L_∞ replaced by $\tilde{L}$. We see as in the proof of Theorem 6.5 that $(\tilde{W}, \tilde{W}^-) \times T^{2N}$ is an admissible pair for Φ and K provided R and R_0 are large enough. Since $M^-(\tilde{L}|_{E_n}) = d_n$, $C^{q+d_n}(\tilde{I}, 0) = \delta_0^q \mathbb{F}$ for all n and therefore $H^q_{\mathcal{E}}(\tilde{W}, \tilde{W}^-) = C^q_{\mathcal{E}}(\tilde{I}, 0) = \delta_0^q \mathbb{F}$. By Künneth's formula [17, p. 8],

$$H^*_{\mathcal{E}}((\tilde{W}, \tilde{W}^-) \times T^{2N}) = H^*_{\mathcal{E}}(\tilde{W}, \tilde{W}^-) \otimes H^*(T^{2N}) = H^*(T^{2N}).$$

Since $H^*(T^{2N}) = H^*(S^1) \otimes \cdots \otimes H^*(S^1)$ ($2N$ times), $H^q(T^{2N})$ is the direct sum of $\binom{2N}{q}$ copies of $\mathbb{F}$ if $0 \le q \le 2N$ and is 0 otherwise (cf. [17, p. 6]). Thus

$$p^q_{\mathcal{E}} = \dim_{\mathcal{E}} H^q_{\mathcal{E}}((\tilde{W}, \tilde{W}^-) \times T^{2N}) = \binom{2N}{q}, \quad 0 \le q \le 2N$$

and $p^q_{\mathcal{E}} = 0$, $q \notin [0, 2N]$. As the coefficients of $Q_{\mathcal{E}}$ are in $[\mathbb{N}_0]$, it follows from Theorem 5.7 that $m^q_{\mathcal{E}} \ge p^q_{\mathcal{E}}$. Denoting the cardinality of the critical set K by $\#K$ and using Theorem 5.8 we obtain

$$\#K = \sum_{q\in\mathbb{Z}} m^q_{\mathcal{E}} \ge \sum_{q\in\mathbb{Z}} p^q_{\mathcal{E}} = \sum_{q=0}^{2N} \binom{2N}{q} = 2^{2N}.$$

□

6.2 Concluding remarks

In the preceding subsection we have assumed that A_∞ and A_0 are constant matrices which satisfy the nonresonance condition $\sigma(JA_\infty) \cap i\mathbb{Z} = \sigma(JA_0) \cap i\mathbb{Z} = \emptyset$. More generally, one can admit t-dependent matrices with 2π-periodic entries and replace the nonresonance condition by certain conditions on G_∞ and G_0. However, the proofs become more technical. See e.g. [30, 36, 51].

Existence of multiple periodic solutions in the setting of Theorem 6.7 has also been studied under the assumption that H is periodic in some (but not necessarily all) z-variables, see e.g. [17, 36, 49]. The result of Conley and Zehnder [21] described in Theorem 6.7 was a starting point for Floer's work on Arnold's conjectures and on what became known as the Floer homology and cohomology, see e.g. the already mentioned references [40, 46], the papers [26, 37] for a solution of the Arnold conjecture, and the recent survey [15].

In [50] the Morse theory of [49] has been applied in order to study bifurcation of nonconstant periodic solutions of small amplitude for the autonomous Hamiltonian system

$\dot{z} = JH'(z)$, $z \in \mathbb{R}^{2N}$. By a change of the independent variable one can equivalently look for bifurcation of 2π-periodic solutions for the system $\dot{z} = \lambda H'(z)$, and the results of [50] assert that if $\Phi_\lambda(z) := \frac{1}{2}\int_0^{2\pi}(-J\dot{z}\cdot z)\,dt - \lambda\int_0^{2\pi} H(z,t)\,dt$ and the Morse index $M_{\mathcal{E}}^-(\Phi_\lambda''(0))$ changes as λ crosses λ_0, then $(0,\lambda_0)$ is a bifurcation point (this can be translated into a statement concerning the original system and the change of index may be expressed in terms of the properties of the matrix $H''(0)$). See also [7] where a more precise result has been obtained by employing a finite-dimensional reduction and equivariant Conley index theory. The above result is local. It has been shown in [22] that if a suitable S^1-degree changes at λ_0, then the bifurcation from $(0,\lambda_0)$ is in fact global.

Other problems where critical point theory for strongly indefinite functionals has been employed are the wave equation of vibrating string type, the beam equation and certain elliptic systems of partial differential equations in bounded domains. For more information on these problems we respectively refer to [36, 58], [58, 59], [36, 50, 59] and the references therein.

If H is convex in z or $H(z,t) = \frac{1}{2}Az\cdot z + G(z,t)$, where G is convex in z, it is possible to replace Φ by a dual functional to which the Morse theory of sections 2 and 3 can be applied. See e.g. [24, 25, 38, 39, 55]. In particular, this approach has turned out to be successful when studying the number of geometrically distinct periodic solutions for autonomous Hamiltonian systems on a prescribed energy surface $H(z) = c$ bounding a convex set in $\mathbb{R}^{2N}$.

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