Some critical minimization problems for functions of bounded variations

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Abstract

Using a new elementary method, we prove the existence of minimizers for various critical problems in $BV(\Omega)$ and also in $W^{k,p}(\Omega)$, $1 < p < \infty$.

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1. Introduction

After the classical results due to Brezis and Nirenberg ([2]), many papers were devoted to critical minimization problems on $W^{1,p}(\Omega)$ $(1 < p < \infty)$ or on some subspaces. See e.g. the list of references in [10].

When $p = 1$, it is necessary to replace $W^{1,1}(\Omega)$ by $BV(\Omega)$, the space of integrable functions with bounded variations on Ω. We know only 3 papers devoted to critical minimization problems on $BV(\Omega)$: [1], [5] and [7]. The critical trace problem in $BV(\Omega)$, treated in [8], is different since it is convex. We exclude this problem.

The existence of optimal functions for the sharp Poincaré inequality

$$c\left\|u - \frac{1}{m(\Omega)}\int_\Omega u\,dx\right\|_{L^{N/(N-1)}(\Omega)} \leq ||Du||_\Omega$$

is proved in [5] when Ω is a ball or a sphere and in [1] when Ω is a bounded domain with $\mathcal{C}^2$ boundary. The proof in [5] uses a specific isoperimetric inequality and in [1] the concentration-compactness principle in $BV(\Omega)$. When $\Omega \subset \mathbb{R}^2$, the results in [1] solve a problem of [3].

The minimization problem

$$\begin{cases} \inf\left[||Du||_\Omega + \displaystyle\int_\Omega a|u|dx + \int_{\partial\Omega}|u|d\sigma\right] \\ \\ u \in BV(\Omega), ||u||_{L^{N/(N-1)}(\Omega)} = 1, \end{cases}$$

is treated in [7] using approximation by subcritical problems and the concentration-compactness principle in $BV(\Omega)$. The penalization term $\displaystyle\int_{\partial\Omega}|u|d\sigma$ replaces the Dirichlet boundary condition (see [7] and [11]). See also [6] and [15] for the existence of critical points.

A general existence theorem for subcritical minimization problems on $BV(\Omega)$ is contained in [11].

In this paper, we solve critical minimization problems on $BV(\Omega)$ by using a new elementary lemma (Lemma 3.2) or a variant (Lemma 4.1). This method is also applicable to critical minimization problems on $W^{1,p}(\Omega)$ $(1 < p < \infty)$ (see Lemma 5.1), is rather simple and avoids any concentration-compactness type argument.

In section 2 we recall some basic properties of functions of bounded variations (see [10] and [16]).

2. Functions of bounded variations

Let Ω be an open subset of $\mathbb{R}^N$. The *variation* of $u \in L^1_{\rm loc}(\Omega)$ is defined by

$$||Du||_\Omega = \sup\left\{\int_\Omega u \,\mathrm{div}\, v\, dx : v \in \mathcal{D}(\Omega, \mathbb{R}^N), ||v||_\infty \leq 1\right\}$$

where

$$||v||_\infty = \sup_{x\in\Omega}\left(\sum_{k=1}^N (v_k(x))^2\right)^{1/2}.$$

The variation is lower semi-continuous.

$$u_n \overset{L^1_{\rm loc}(\Omega)}{\longrightarrow} u \Rightarrow ||Du||_\Omega \leq \underline{\lim}_{n\to\infty} ||Du_n||_\Omega.$$

On

$$BV(\Omega) = \{u \in L^1(\Omega) : ||Du||_\Omega < \infty\}$$

we define the norm

$$||u||_{BV(\Omega)} = ||Du||_\Omega + ||u||_{L^1(\Omega)}$$

and the distance of *strict convergence*

$$d(f,g) = \Big| ||Df||_\Omega - ||Dg||_\Omega \Big| + ||f-g||_{L^1(\Omega)}.$$

The sequence (u_n) *converges weakly* to u in $BV(\Omega)$ (written $u_n \rightharpoonup u$) if

$$||u_n - u||_{L^1} \to 0, \quad n\to\infty,$$

$$\partial_k u_n \rightharpoonup \partial_k u \text{ in } [\mathcal{C}_0(\Omega)]^*, \quad n\to\infty, 1 \leq k \leq N.$$

where $[C_0(\Omega)]^*$ denotes the space of finite measures on Ω.

It is clear that

norm convergence $\Rightarrow$ strict convergence $\Rightarrow$ weak convergence.

We now assume that Ω is a bounded domain of $\mathbb{R}^N$ ($N \geq 2$) with Lipschitz boundary. Let us recall (see [10]) that, for every $u \in BV(\Omega)$, the *trace* of u, $\gamma_0(u)$, belongs to $L^1(\partial\Omega)$ and that the *extension by* 0

$$\begin{cases} u_0(x) & = u(x), \quad x \in \Omega, \\ & = 0, \qquad x \in \mathbb{R}^N \backslash \{0\}, \end{cases}$$

belongs to $BV(\mathbb{R}^N)$. Moreover,

$$||Du_0||_{\mathbb{R}^N} = ||Du||_\Omega + \int_{\partial\Omega} |\gamma_0(u)| d\sigma$$

defines an equivalent norm on $BV(\Omega)$. The space $W^{1,1}(\Omega)$ is dense in $BV(\Omega)$ with respect to the strict convergence (not the norm convergence!) and the trace operator $\gamma_0 : BV(\Omega) \to L^1(\partial\Omega)$ is continuous with respect to the strict convergence (not the weak convergence!).

We will also denote by u the trace of u and the extension of u by 0.

Let us denote by 1^* the *critical exponent* $N/(N-1)$ and by V_N the volume of the unit ball in $\mathbb{R}^N$. The following inequality is due to Cherrier [4].

Theorem 2.1. *For every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that, for all $u \in BV(\Omega)$,*

$$(N(V_N/2)^{1/N} - \varepsilon)||u||_{L^{1^*}(\Omega)} \leq ||Du||_\Omega + c_\varepsilon ||u||_{L^1(\Omega)}.$$

Let us recall that, for $1 \leq p < 1^*$, the imbedding $BV(\Omega) \subset L^p(\Omega)$ is compact and that the imbedding $BV(\Omega) \subset L^{1^*}(\Omega)$ is continuous (but not compact!).

We will also need the sharp Gagliardo-Nirenberg inequality due to Mazýa and Federer and Fleming (see [14]):

Theorem 2.2. *For every $u \in L^{1^*}(\mathbb{R}^N)$,*

$$NV_N^{1/N} ||u||_{L^{1^*}(\mathbb{R}^N)} \leq ||Du||_{\mathbb{R}^N}.$$

Moreover equality holds if and only if u is the characteristic function of a ball.

We will use *truncation* as a basic tool. We define, for $h > 0$,

$$T_h(s) = \min(\max(s, -h), h), R_h(s) = s - T_h(s).$$

Proposition 2.3. *For every $u \in BV(\Omega)$,*

$$||Du||_\Omega = ||DT_h u||_\Omega + ||DR_h u||_\Omega.$$

Proof. It is clear that

$$||Du||_\Omega \leq ||DT_h u||_\Omega + ||DR_h u||_\Omega.$$

Let $(u_n) \subset W^{1,1}(\Omega)$ be such that $u_n \to u$ strictly in $BV(\Omega)$. Then, by lower semicontinuity,

$$\begin{aligned} ||DT_h u||_\Omega + ||DR_h u||_\Omega &\leq \varliminf_{n\to\infty} ||\nabla T_h u_n||_{L^1(\Omega)} + \varliminf_{n\to\infty} ||\nabla R_h u_n||_{L^1(\Omega)} \\ &\leq \lim_{n\to\infty} ||\nabla u_n||_{L^1(\Omega)} = ||Du||_\Omega. \end{aligned}$$ □

The proof of Proposition 2.3 was communicated to us by J. Van Schaftingen.

3. Critical minimizations problems in $BV(\Omega)$

The following result is due to Degiovanni and Magrone in the case $p = 1^*$ (see [6] p. 603). We give the proof for the sake of completeness.

Lemma 3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ and let $1 \leq p < \infty$ and $(u_n) \subset L^p(\Omega)$ be such that*

a) $\sup ||u_n||_p < \infty$

b) (u_n) *converges to u almost everywhere on Ω.*

Then

$$\lim_{n\to\infty}(||u_n||_p^p - ||R_h u_n||_p^p) = (||u||_p^p - ||R_h u||_p^p).$$

Proof. Let us define

$$f(s) = |s|^p - |R_h(s)|^p.$$

For every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(s) - f(t)| \leq \varepsilon(|s|^p + |t|^p) + C_\varepsilon.$$

It follows from Fatou's lemma that

$$\begin{aligned}
&2\varepsilon\int_\Omega |u|^p dx + C_\varepsilon m(\Omega)\\
\leq\ &\varliminf_{n\to\infty}\int_\Omega \varepsilon(|u_n|^p + |u|^p) + C_\varepsilon - |f(u_n) - f(u)|dx\\
\leq\ &\varepsilon\sup_n\int_\Omega |u_n|^p dx + \varepsilon\int_\Omega |u|^p dx + C_\varepsilon m(\Omega) - \varlimsup_{n\to\infty}\int_\Omega |f(u_n) - f(u)|dx.
\end{aligned}$$

Hence

$$\varlimsup_{n\to\infty}\int_\Omega |f(u_n) - f(u)|dx \leq \varepsilon\sup_n\int_\Omega |u_n|^p dx.$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete. $\square$

In this section, we assume that Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary.

Lemma 3.2. *Let $a \in \mathcal{C}(\bar{\Omega})$ and $b \in \mathcal{C}(\partial\Omega)$ be such that φ defined on $BV(\Omega)$ by*

$$\varphi(u) = ||Du||_\Omega + \int_\Omega a|u|dx + \int_{\partial\Omega} b|u|d\sigma$$

satisfies

$$c = \inf\{\varphi(u)/||u||_{L^{1^*}(\Omega)} : u \in BV(\Omega)\backslash\{0\}\} > 0.$$

Let $(u_n) \subset BV(\Omega)$ be such that $||u_n||_{L^{1^*}(\Omega)} = 1$, $\varphi(u_n) \to c$, $n \to \infty$, and $u_n \rightharpoonup u$ in $BV(\Omega)$. Then either $||u||_{L^{1^*}(\Omega)} = 0$ or $||u||_{L^{1^*}(\Omega)} = 1$.

Proof. By going if necessary to a subsequence, we can assume that $u_n \to u$ a.e. on Ω. We have, using the preceding lemma,

$$\begin{aligned} c &= \lim_{n\to\infty} [\varphi(T_h u_n) + \varphi(R_h u_n)] \\ &\geq c \lim_{n\to\infty} [||T_h u_n||_{1^*} + ||R_h u_n||_{1^*}] \\ &= c\big[||T_h u||_{1^*} + (1 + ||R_h u||_{1^*}^{1^*} - ||u||_{1^*}^{1^*})^{1/1^*}\big]. \end{aligned}$$

When $h \to \infty$, we obtain

$$1 \geq (||u||_{1^*}^{1^*})^{1/1^*} + (1 - ||u||_{1^*}^{1^*})^{1/1^*},$$

so that $||u||_{1^*} = 0$ or $||u||_{1^*} = 1$. $\square$

We consider first the case when $b = 0$. We assume that $a \in \mathcal{C}(\bar{\Omega})$ and

(A1) $\quad 0 < S_0(a,\Omega) = \inf \left\{ \left[||Du||_\Omega + \int_\Omega a|u|dx \right] /||u||_{L^{1^*}(\Omega)} : u \in BV(\Omega)\backslash\{0\} \right\},$

(A2) $\quad S_0(A,\Omega) < N(V_N/2)^{1/N}$.

Theorem 3.3. *Under assumptions* (A1), (A2), *there exists* $u \in BV(\Omega)\backslash\{0\}$ *such that* $u \geq 0$ *and*

$$S_0(a,\Omega)||u||_{L^{1^*}(\Omega)} = ||Du||_\Omega + \int_\Omega a|u|dx.$$

Proof. Let $(u_n) \subset BV(\Omega)$ be such that $||u_n||_{1^*} = 1$ and

$$||Du_n||_\Omega + \int_\Omega a|u_n|dx \to S_0(a,\Omega).$$

Since (u_n) is bounded in $BV(\Omega)$, we can assume that $u_n \rightharpoonup u$ in $BV(\Omega)$. Let $0 < \varepsilon < N(V_N/2)^{1/N} - S_0(a,\Omega)$. It follows from Theorem 2.1 that, for some $c_\varepsilon > 0$,

$$\begin{aligned} S_0(a,\Omega) &= \lim_{n\to\infty} \left[||Du_n||_\Omega + \int_\Omega a|u_n|dx \right] \\ &\geq N(V_N/2)^{1/N} - \varepsilon - c_\varepsilon \int_\Omega |u|dx + \int_\Omega a|u|dx. \end{aligned}$$

Hence, $u \neq 0$. The preceding lemma implies that $||u||_{1^*} = 1$. Since, by lower semi-continuity,

$$||Du||_\Omega + \int_\Omega a|u|dx \leq S_0(a,\Omega),$$

u is a minimizer for $S_0(a,\Omega)$. Since $||D|u|||_\Omega \leq ||Du||_\Omega$, we can replace u by $|u|$. $\square$

The following result gives a concrete sufficient condition for (A2).

Theorem 3.4. *Let Ω be a bounded domain with $\mathcal{C}^2$ boundary and let $a \in \mathcal{C}(\bar{\Omega})$ be such that, for some $y \in \partial\Omega$,*

$$2\frac{N-1}{N+1}\frac{V_{N-1}}{V_N}H(y) > a(y),$$

where $H(y)$ denotes the mean curvature of $\partial\Omega$ at y. Then (A2) *is satisfied.*

Proof. We can assume that $y = 0$. For $r > 0$ small enough, we have

$$\delta = \frac{N-1}{N+1}V_{N-1}H(0) - \frac{V_N}{2}A > 0,$$

where $A = \sup\{a(x) : x \in \Omega \cap B(0,r)\}$. Let us define $u_\varepsilon = \chi_{\Omega\cap B(0,\varepsilon)}$. By formula (1) in [12], we have, for $\varepsilon \to 0$,

$$\begin{aligned}
||u_\varepsilon||_{1^*}^{1^*} &= m(\Omega \cap B(0,\varepsilon)) = \frac{V_N}{2}\varepsilon^N - \frac{N-1}{N+1}\frac{V_{N-1}}{2}H(0)\varepsilon^{N+1} + o(\varepsilon^{N+1}),\\
||Du_\varepsilon||_\Omega &= N\frac{V_N}{2}\varepsilon^{N-1} - (N-1)\frac{V_{N-1}}{2}H(0)\varepsilon^N + o(\varepsilon^N),\\
\int_\Omega a\, u_\varepsilon dx &\le A\frac{V_N}{2}\varepsilon^N + o(\varepsilon^N).
\end{aligned}$$

It follows that, for $\varepsilon \to 0$,

$$\begin{aligned}
&\left[||Du_\varepsilon||_\Omega + \int_\Omega a\, u_\varepsilon dx\right] / ||u_\varepsilon||_{1^*}\\
&\le \left(\frac{V_N}{2}\right)^{\frac{1-N}{N}} \left[N\frac{V_N}{2} - \frac{N-1}{N+1}V_{N-1}H(0)\varepsilon + \frac{V_N}{2}A\varepsilon\right] + o(\varepsilon)\\
&= N(V_N/2)^{1/N} - \delta(V_N/2)^{\frac{1-N}{N}}\varepsilon + o(\varepsilon),
\end{aligned}$$

so that $S_0(a,\Omega) < N(V_N/2)^{1/N}$. $\square$

We consider now the case when $b = 1$. The following result is due to Demengel [7], but our proof, using Lemma 3.1, is simpler.

Let us recall that, for $u \in BV(\Omega)$,

$$||Du||_{\mathbb{R}^N} = ||Du||_\Omega + \int_{\partial\Omega} |u| d\sigma.$$

We assume that $a \in \mathcal{C}(\bar{\Omega})$ and

(B1) $\quad 0 < S_1(a,\Omega) = \inf\left\{\left[||Du||_{\mathbb{R}^N} + \int_\Omega a|u|dx\right]/||u||_{L^{1^*}(\Omega)} : u \in BV(\Omega)\backslash\{0\}\right\},$

(B2) $\quad S_1(a,\Omega) < NV_N^{1/N}.$

Theorem 3.5. *Under assumptions* (B1), (B2), *there exists* $u \in BV(\Omega)\backslash\{0\}$ *such that* $u \geq 0$ *and*

$$S_1(a,\Omega)||u||_{L^{1^*}(\Omega)} = ||Du||_{\mathbb{R}^N} + \int_\Omega a|u|dx.$$

Proof. Let $(u_n) \subset BV(\Omega)$ be such that $||u_n||_{1^*} = 1$ and

$$||Du_n||_{\mathbb{R}^N} + \int_\Omega a|u_n|dx \to S_1(a,\Omega).$$

The sequence (u_n) is bounded in $BV(\mathbb{R}^N)$. We can assume that $u_n \rightharpoonup u$ in $BV(\mathbb{R}^N)$. It follows from Theorem 2.2 that

$$\begin{aligned} S_1(a,\Omega) &= \lim_{n\to\infty}\left[||Du||_{\mathbb{R}^N} + \int_\Omega a|u_n|dx\right] \\ &\geq NV_N^{1/N} + \int_\Omega a|u|dx. \end{aligned}$$

By assumption (B2), $u \neq 0$. Lemma 3.2 implies that $||u||_{1^*} = 1$. Since, by lower semi-continuity,

$$||Du||_{\mathbb{R}^N} + \int_\Omega a|u|dx \leq S_1(a,\Omega),$$

u is a minimizer for $S_1(a,\Omega)$. Since $||D|u|||_{\mathbb{R}^N} \leq ||Du||_{\mathbb{R}^N}$, we can replace u by $|u|$. □

4. Poincaré inequality

Let us recall the general Poincaré inequality in $BV(\Omega)$ due to Meyers and Ziemer [13].

Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary and let $f \in L^N(\Omega)$ be such that $\int_\Omega f\,dx = 1$. Then

$$S_2(f,\Omega) = \inf\left\{||Du||_\Omega/||u||_{L^{1^*}(\Omega)} : u \in BV(\Omega)\backslash\{0\}, \int_\Omega fu\,dx = 0\right\} > 0.$$

When $f \equiv 1/m(\Omega)$, this is the Poincaré inequality.

Lemma 4.1. *Let* $f \in L^N(\Omega)$ *be such that* $\int_\Omega f\,dx = 1$ *and let* $(u_n) \subset BV(\Omega)$ *be such that* $||u_n||_{L^{1^*}(\Omega)} = 1$, $\int_\Omega fu_n dx = 0$, $||Du_n||_\Omega \to S_2(f,\Omega)$, $n \to \infty$, *and* $u_n \rightharpoonup u$ *in* $BV(\Omega)$. *Then either* $||u||_{L^{1^*}(\Omega)} = 0$ *or* $||u||_{L^{1^*}(\Omega)} = 1$.

Proof. By going if necessary to a subsequence, we can assume that $u_n \to u$ a.e. on Ω. Let us define, for $h > 0$ and $n \in \mathbb{N}$,

$$c_{h,n} = \int_\Omega f\, T_h u_n dx, \quad c_h = \int_\Omega f\, T_h u\, dx.$$

Using Lemma 3.1, we obtain,

$$\begin{aligned} S_2(f,\Omega) &= \lim_{n\to\infty} [||DT_h u_n||_\Omega + ||DR_h u_n||_\Omega] \\ &\geq S_2(f,\Omega) \varliminf_{n\to\infty} [||T_h u_n - c_{h,n}||_{1^*} + ||R_h u_n + c_{h,n}||_{1^*}] \\ &\geq S_2(f,\Omega) \lim_{n\to\infty} [||T_h u_n||_{1^*} + ||R_h u_n||_{1^*} - 2||c_{h,n}||_{1^*}] \\ &= S_2(f,\Omega) \left[||T_h u||_{1^*} + \left[1 + ||R_h u||_{1^*}^{1^*} - ||u||_{1^*}^{1^*}\right]^{1/1^*} - 2||c_h||_{1^*}\right]. \end{aligned}$$

Since $\lim_{h\to\infty} c_h = \lim_{h\to\infty} \int_\Omega f\, T_h u\, dx = \int_\Omega f u\, dx = 0$, we have

$$1 \geq \left[||u||_{1^*}^{1^*}\right]^{1/1^*} + \left[1 - ||u||_{1^*}^{1^*}\right]^{1/1^*},$$

so that $||u||_{1^*} = 0$ or $||u||_{1^*} = 1$. □

The following theorem was proved by Bouchez and Van Schaftingen in the case $f \equiv 1/m(\Omega)$ (see [1]).

Theorem 4.2. *Let Ω be a bounded domain of $\mathbb{R}^N$ with $\mathcal{C}^2$ boundary and let $f \in L^N(\Omega)$ be such that $\int_\Omega f\, dx = 1$. Then there exists $u \in BV(\Omega)\backslash\{0\}$ such that $\int_\Omega f\, u\, dx = 0$ and*

$$S_2(f,\Omega)||u||_{L^{1^*}(\Omega)} = ||Du||_\Omega.$$

Proof. 1) Let us first prove that

$$(*) \qquad S_2(f,\Omega) < N(V_N/2)^{1/N}.$$

We can assume that $0 \in \partial\Omega$ and that $H(0)$, the mean curvature of $\partial\Omega$ at 0, is positive. Let us define, as in [1], for $\varepsilon > 0$ small enough,

$$\begin{aligned} u_\varepsilon &= \chi_{\Omega\cap B(0,\varepsilon)} - \lambda_\varepsilon \chi_{\Omega\backslash B(0,\varepsilon)}, \\ \lambda_\varepsilon &= \int_{\Omega\cap B(0,\varepsilon)} f\, dx \Big/ \int_{\Omega\backslash B(0,\varepsilon)} f\, dx. \end{aligned}$$

Hölder inequality implies that $\lambda_\varepsilon = o(\varepsilon^{N-1})$. By formula (1) in [12], we have, for $\varepsilon \to 0$,

$$\begin{aligned} ||u_\varepsilon||_{1^*}^{1^*} &\geq m(\Omega \cap B(0,\varepsilon)) = \frac{V_N}{2}\varepsilon^N - \frac{N-1}{N+1}\frac{V_{N-1}}{2}H(0)\varepsilon^{N+1} + o(\varepsilon^{N+1}), \\ ||Du_\varepsilon||_\Omega &= (1+\lambda_\varepsilon)||D\chi_{\Omega\cap B(0,\varepsilon)}|| \\ &= N\frac{V_N}{2}\varepsilon^{N-1} - (N-1)\frac{V_{N-1}}{2}H(0)\varepsilon^N + o(\varepsilon^N). \end{aligned}$$

It follows that, for $\varepsilon \to 0$,

$$||Du_\varepsilon||_\Omega / ||u_\varepsilon||_{1^*} \leq \left(\frac{V_N}{2}\right)^{\frac{1-N}{N}} \left[N\frac{V_N}{2} - \frac{N-1}{N+1}V_{N-1}H(0)\varepsilon + o(\varepsilon)\right],$$

so that $(*)$ is satisfied.

2) Let $(u_n) \subset BV(\Omega)$ be such that $||u_n||_{1^*} = 1$, $\int_\Omega f\, u_n dx = 0$ and

$$||Du_n||_\Omega \to S_2(f,\Omega), \quad n \to \infty.$$

We can assume that $u_n \rightharpoonup u$ in $BV(\Omega)$. Let $0 < \varepsilon < N(V_N/2)^{1/N} - S_2(f,\Omega)$. It follows from Theorem 2.1 that, for some $c_\varepsilon > 0$,

$$S_2(f,\Omega) = \lim_{n\to\infty} ||Du_n||_\Omega \geq N(V_N/2)^{1/N} - \varepsilon - c_\varepsilon \int_\Omega |u| dx.$$

Hence $u \neq 0$. The preceding lemma implies that $||u||_{1^*} = 1$. Since $\int_\Omega f\, u\, dx = 0$ and, by lower semi-continuity,

$$||Du||_\Omega \leq S_2(f,\Omega),$$

u is a minimizer for $S_2(f,\Omega)$. □

5. Critical minimization problems in $W^{1,p}(\Omega)$

In this section, we assume that Ω is a smooth bounded domain of $\mathbb{R}^N$. We define, for $1 < p < \infty$, the critical exponent $p^* = Np/(N-p)$ and

$$\begin{aligned} X_0 &= W^{1,p}(\Omega), \\ X_1 &= W_0^{1,p}(\Omega), \\ X_2 &= \left\{u \in W^{1,p}(\Omega) : \int_\Omega f\, u\, dx = 0\right\} \end{aligned}$$

where $f \in L^{p^*}(\Omega)$ and $\int_\Omega f\, dx = 1$.

The following lemma is a variant of Lemma 3.2 and Lemma 4.1 with a similar proof.

Lemma 5.1. *Let $a \in \mathcal{C}(\bar{\Omega})$ be such that φ defined on X_j (where $j = 0, 1$ or 2) by*

$$\varphi(u) = \int_\Omega |\nabla u|^p dx + \int_\Omega a|u|^p dx$$

satisfies

$$c_j = \inf \left\{ \varphi(u)/||u||^p_{L^{p^*}(\Omega)} : u \in X_j \backslash \{0\} \right\} > 0.$$

Let $(u_n) \subset X_j$ be such that $||u_n||_{L^{p^*}(\Omega)} = 1$, $\varphi(u_n) \to c_j$, $n \to \infty$, and $u_n \rightharpoonup u$ in X_j. Then either $||u||_{L^{p^*}(\Omega)} = 0$ or $||u||_{L^{p^*}(\Omega)} = 1$.

The preceding lemma is applicable to many quasilinear critical problems as considered e.g. in [9].

Let us define

$$S(p, \mathbb{R}^N) = \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^p dx/||u||^p_{L^{p^*}(\mathbb{R}^N)} : u \in \mathcal{D}(\mathbb{R}^N)\backslash\{0\} \right\}.$$

The following Theorem is a variant of Theorems 3.3, 3.5 and 4.2.

Theorem 5.2. a) *If $0 < c_0 < S(p, \mathbb{R}^N)/2^{p/N}$, then c_0 is achieved.*

b) *If $0 < c_1 < S(p, \mathbb{R}^N)$, then c_1 is achieved.*

c) *If $0 < c_2 < S(p, \mathbb{R}^N)/2^{p/N}$, then c_2 is achieved.*

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