“Quantum pion liquid” in QCD at $\mu \neq 0$? A tale in two acts

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Lunch Club Seminar, Institut für Theoretische Physik, JLU Giessen

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DFG

CRC-TR 211
Strong-interaction matter under extreme conditions

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Outline

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1. Introduction: Why are phases with spatial modulations relevant in QCD at $\mu \neq 0$?

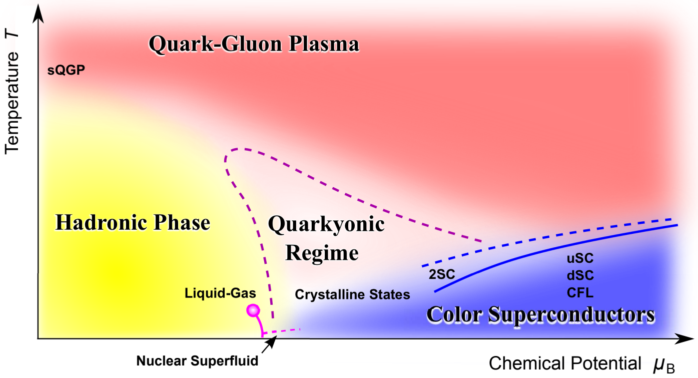
2. Quantum pion liquid from mixing effects: A study based on four-fermion models
   - → MW, PRD **110**, 034008, arXiv: 2403.07430
   - → Pannullo, MW, PRD **108**, 036011, arXiv: 2305.09444

3. Scalar $\mathrm{O}(N)$ model: Disordering of inhomogeneous condensates from quantum fluctuations
   - → MW, Valgushev, arXiv: 2403.18604

4. Experimental signature and conclusions
   - → Nussinov, Ogilvie, Pannullo, Pisarski, Rennecke, Schindler, MW, arXiv: 2410.22418

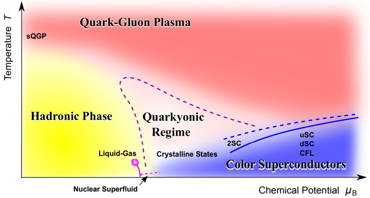
Introduction: Phases with spatial modulations in QCD at $\mu \neq 0$

- QCD Phase diagram in $T - \mu_B$ plane: A lot of open questions
  - Moat regimes, inhomogeneous chiral phases, quantum pion liquid: Spatial modulations of the order parameter?



[Fukushima, Sasaki, Prog. Part. Nucl. Phys. **72** (2013), arXiv: 1301.6377]

Two arguments for spatially modulated regimes



[Fukushima, Sasaki, Prog. Part. Nucl. Phys. **72** (2013), arXiv: 1301.6377]

Why are phases with spatial modulations relevant: Two key arguments

- Indications from the Functional renormalization group
  - Fu, Pawlowski, Rennecke, PRD **101**, 054032 (2020), arXiv: 1909.02991
  - Fu, Pawlowski, Pisarski, Rennecke, Wen, Yin, arXiv: 2412:15949
- Often overlooked symmetry pattern for QCD when $\mu \neq 0$
  - Meisinger, Ogilvie, Phil. Trans. Roy. Soc. Lond. A 371, 20120058 (2013), arXiv: 1208.5077
  - Nishimura, Ogilvie, Pangeni, PRD **90**, 045039 (2014) & PRD **91**, 054004 (2015)

Generalized $\mathcal{PT}$-symmetry in QCD at $\mu \neq 0$

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$\mathcal{CK}$ symmetry at $\mu \neq 0$

- Charge conjugation in Euclidean spacetime $A_\mu \to -A_\mu^T$; Wilson loop $W \Rightarrow W^\dagger$
- At $\mu = 0$: Fermion determinant can be expanded in Wilson loops $\mathrm{tr}_F W$ where $\mathrm{tr}_F W$ and $\mathrm{tr}_F W^\dagger$ appear with similar coefficients $\Rightarrow \ln \det[A_\mu] \in \mathbb{R}$
- Charge conjugation $\mathcal{C}$ exchanges $\mathrm{tr}_F W$ and $\mathrm{tr}_F W^\dagger$
- At $\mu \neq 0$: Wilson loops $\mathrm{tr}_F W/W^\dagger$ with non-trivial winding number n are weighted by $\mathrm{e}^{\pm\beta\mu} \Rightarrow$ Broken $\mathcal{C}$ symmetry, complex weights in the path integral
- Invariance of QCD action at $\mu \neq 0$ under $\mathcal{CK}$ operation: $\mathrm{tr}_F W \to \mathrm{tr}_F W^T = \mathrm{tr}_F W$ ($\mathcal{K}$ is complex conjugation). This is a $\mathcal{PT}$-type symmetry!

Here: $\mathcal{P}$: Discrete, linear symmetry; $\mathcal{T}$: Discrete, antilinear symmetry

[Meisinger, Ogilvie, Phil. Trans. Roy. Soc. Lond. A 371, 20120058 (2013), arXiv: 1208.5077]

Generalized $\mathcal{PT}$-symmetry: Implications on the Hessian matrix

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- Consider a model with homogeneous (thermal) vev's $\langle\vec{\phi}\rangle$ and propagators G_{ϕ_j}
- Low momentum representation of (scalar) propagators with Mass / Hessian matrix $\mathcal{M}$

$$G^{-1}(q^2) = q^2 + \mathcal{M}$$

[Schindler, Schindler, Medina, Ogilvie, PRD **102**, 114510 (2020)]

[Schindler, Schindler, Ogilvie, J. Phys. Conf. Ser. 2038 (2021)]

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- Conventional, hermitian mass matrix with eigenvalues $\lambda_j \in \mathbb{R} \Rightarrow$ usual exponential decay

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- With $\mathcal{PT}$-type symmetry: $\mathcal{M}$ and $\mathcal{M}^*$ with same EVs $\lambda_j \in \mathbb{C}$

$$\mathcal{M} = \Sigma\mathcal{M}^*\Sigma \Rightarrow \text{Compl. conj. EV pairs}$$

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What are the implications?

[Schindler, Schindler, Medina, Ogilvie, PRD **102**, 114510 (2020)]

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Generalized $\mathcal{PT}$-symmetry: Implications on the phase diagram

$$G^{-1}(q^2) = q^2 + \mathcal{M}$$

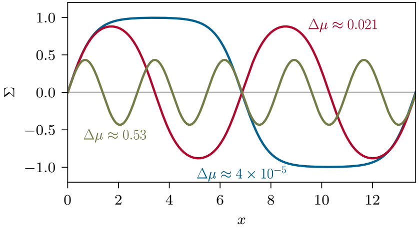
$\mathcal{PT}$-type symmetry: What are the implications?

Eigenvalues E_i of $\mathcal{M}$	**Position-space propagator behavior**	**Region**
All positive	Exponential decay	Normal
Odd number of $E_i < 0$	Exponential growth	Unstable
Some $E_i = E_j^*$	Sinusoidally-modulated exponential	$\mathcal{PT}$ broken
Even number of $E_i < 0$	Homogeneous solution unstable at some $p \neq 0$	Patterned vacuum

[Schindler, Schindler, Ogilvie, PoS LATTICE2021 (2022)]

Inhomogeneous phase (IP)

- "Patterned"
- Long-range order & Translational SSB!
- $\langle\phi_j\rangle \sim \langle\bar{\psi}\Gamma_j\psi\rangle = f_{\mathrm{os}}(\mathbf{x})$
- $\langle\phi(x)\phi(0)\rangle \sim C_{\mathrm{osc}}(x)$



[Buballa, Carignano, PPNP **81**, 39-96 (2015)]

Quantum pion liquid (QπL)

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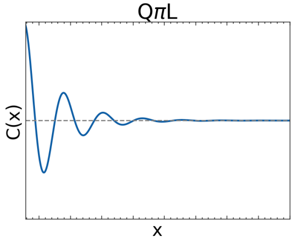
- Disordering through Goldstone modes of chiral SSB?
- $\langle\phi_j\rangle \sim \langle\bar{\psi}\Gamma_j\psi\rangle = \mathrm{const.}$
- $C(x) \sim \mathrm{e}^{-mx} C_{\mathrm{osc}}(x)$



[Pisarski et al., PRD **102**, 016015 (2020)] [MW, Valgushev, arXiv:2403.18640]

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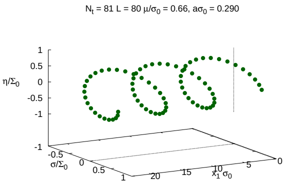

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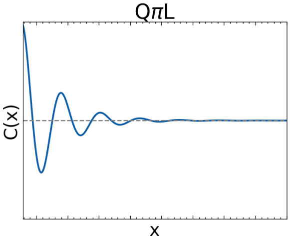

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Exotic phases in finite density QCD: Spatial modulations?

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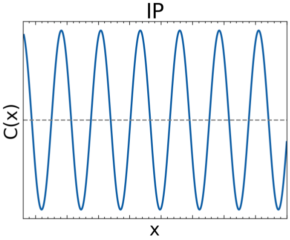

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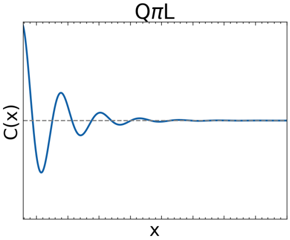

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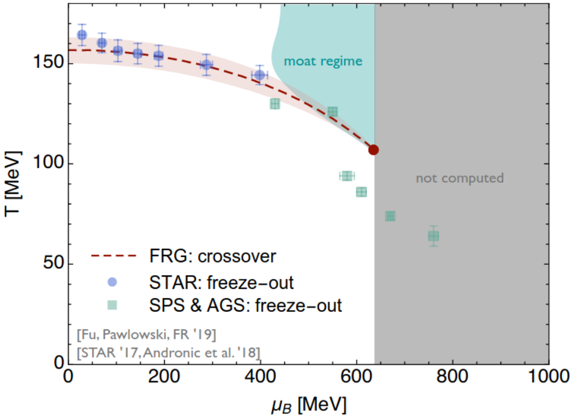

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Exotic phases at $\mu \neq 0$ from functional methods: FRG

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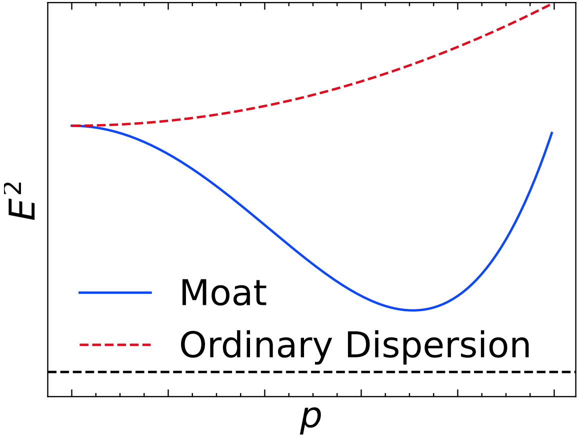

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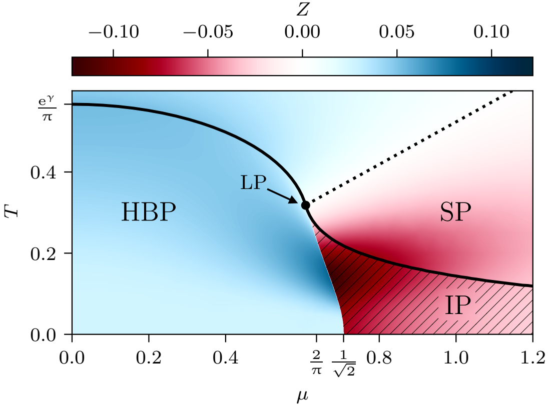
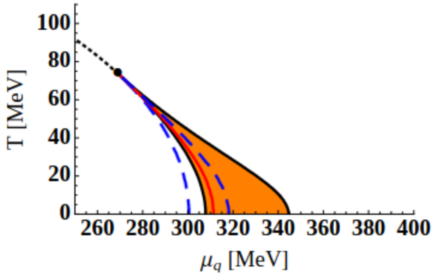
1 + 1-dimensional Four-Fermion model



[Thies, Urlichs, PRD **67**, 125015 (2003)] [Koenigstein, Pannullo, Rechenberger, MW, Steil, (2022)]

- Four-quark interactions / quark-meson as low-energy approximation for QCD with $\Lambda \gtrsim 1\mathrm{GeV}$
- Nambu-Jona-Lasinio-type models / quark-meson models for spontaneous chiral symmetry breaking mechanism; no sign problem !
- IPs appear in phase diagrams of various of those models [Buballa, Carignano, PPNP **81**, 39-96 (2015)]



[Nickel, PRD **80**, 074025 (2009)]

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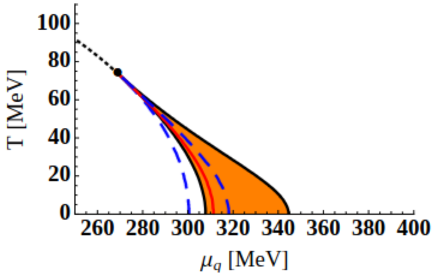

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BUT: Evidence that IPs are cutoff artifacts from multiple studies!

QCD-inspired models: Four-fermion and quark-meson models

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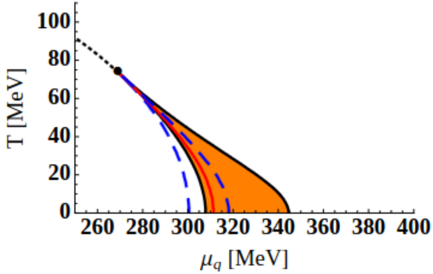

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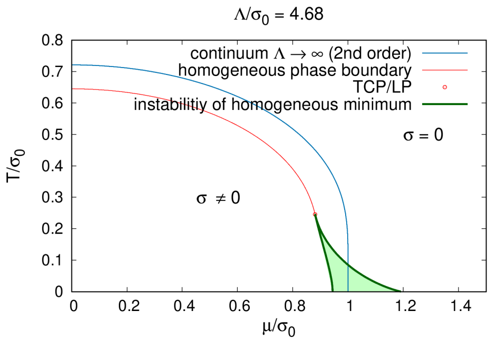
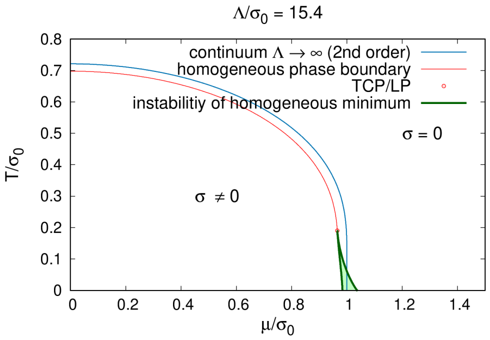
BUT: Evidence that IPs are cutoff artifacts from multiple studies!

[Narayanan, PRD (2021)], [Buballa et al., PRD (2021)], [Pannullo, MW, PRD (2023)], [Pannullo, PRD (2023)], [Koenigstein, Pannullo, PRD (2023)], [Pannullo, MW, Wagner, PRD (2024)]

Regulator dependence of IPs: 2+1-dimensional GN model





- Strong regulator dependence of results & IP vanishes when renormalizing the theory !

[Buballa, Kurth, Wagner, MW, PRD **103**, 034503 (2020), arXiv: 2012.09588]

- No instability towards IP in general four-fermion model with scalar $(S = 0)$ channels

[Pannullo, MW, PRD **108**, 036011 (2023) arXiv:2305.09444]

- Strong regularization scheme dependence in non-renorm. NJL model

[Pannullo, MW, Wagner, PRD **110**, 076007 (2024), arXiv: 2406.11312]

Quantum pion liquid from mixing effects: A study based on four-fermion models

Generic four-fermion model setup

- Study of $d+1$-dimensional models

$$\mathcal{S}_{\mathrm{FF}}[\bar{\psi},\psi] = \int \mathrm{d}^{d+1}x \left\{ \bar{\psi}\left(\partial\!\!\!/ + \gamma_0\mu\right)\psi - \sum_j \left(\frac{\lambda_j}{2N} \left(\bar{\psi}\, c_j\, \psi\right)^2 \right) \right\}$$

- 4×4 Dirac basis for chiral symmetry, 2 flavors ($\gamma_{45} = \mathrm{i}\gamma_4\gamma_5$)
- 32 independent (iso-)spin matrices c_j with $\bar{\psi}c_j\psi$ carrying different quantum numbers
- Bosonized version

$$S[\bar{\psi},\psi,\vec{\phi}] = \int \mathrm{d}^{d+1}x \left\{ N \sum_j \frac{\phi_j^2}{2\lambda_j} + \bar{\psi}Q\psi \right\}, \quad Q = \partial\!\!\!/ + \gamma_0\mu + \sum_j c_j\,\phi_j,$$

- Integrate out $\bar{\psi},\psi \Rightarrow S_{\mathrm{eff}}[\vec{\phi}] \sim \ln \mathrm{Det} Q$

How to detect the IP

Large-N limit / Mean-field approximation: No integration about $\vec{\phi}$

$$\Rightarrow \Omega \sim \min_{\vec{\phi}} S_{\text{eff}}[\vec{\phi}]$$

- Global minimization of $S_{\text{eff}}[\vec{\phi}(\mathbf{x})]$ using ansatzes / lattice field theory

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- Stability analysis of homogeneous ground state $\vec{\phi}(\mathbf{x}) = \vec{\bar{\phi}} = \text{const.}$

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Stability analysis

$$\phi_j(x) = \bar{\phi}_j + \delta\phi_j(\mathbf{x})$$

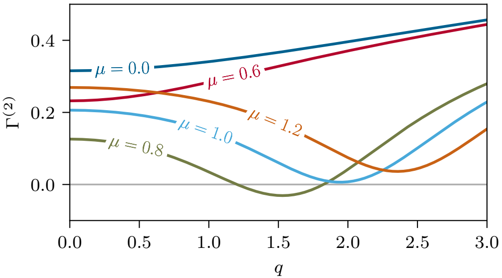
- First non-vanishing correction expressed by Hessian matrix (analog of mass matrix $\mathcal{M}$)

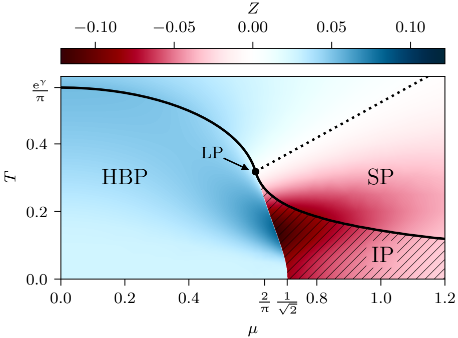
$$\frac{S^{(2)}_{\text{eff}}}{N} = \tfrac{\beta}{2} \int \frac{\mathrm{d}^2 q}{(2\pi)^2}\, \delta\vec{\phi}^{\,T}(-\mathbf{q}) H(|\mathbf{q}|) \delta\vec{\phi}(\mathbf{q}), \quad H_{\phi_j\phi_k}(q) = \left(\tfrac{\delta_{j,k}}{\lambda_j}\right) + \frac{1}{\beta} \sum\!\!\!\!\!\!\int_p \operatorname{tr}\left[S(p + (0,\vec{q}))\, c_j\, S(p)\, c_k\right]$$

- $S \equiv$ bare, free fermion propagator with mass $M^2(\bar{\phi}_j)$ at fixed μ and T
- Eigenvalues of Hessian: Bosonic two-point functions $\Gamma^{(2)}_{\varphi_j}(q) = \left(\langle \varphi_j \varphi_j \rangle_c\right)^{-1}$
- Inhomogeneous phase: $\Gamma^{(2)}(q) < 0$ for $q \neq 0$ Moat: $Z = \frac{\mathrm{d}^2 \Gamma^{(2)}}{\mathrm{d}q^2}(q = 0) < 0$
- Quantum pion liquid: $\Gamma^{(2)}(q = 0) \in \mathbb{C}$ but appear in complex-conjugate pairs!

$$S[\bar\psi,\psi] = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}x \left\{ \bar\psi \left(\partial\!\!\!/ + \gamma_0 \mu\right) \psi - \tfrac{\lambda}{2N} \left(\bar\psi \psi\right)^2 \right\} \xrightarrow{\text{Bosonize}} S[\bar\psi,\psi,\sigma] = \int \mathrm{d}^2 x \left\{ \frac{\sigma^2}{2\lambda} + \bar\psi \left(\partial\!\!\!/ + \gamma_0 \mu + \sigma\right) \psi \right\}$$

$T/\Sigma_0 = 0.15$





[Koenigstein, Pannullo, Rechenberger, MW, Steil, PRD (2022)]

- $\mu = 1.2$ corresponds to a Moat regime
- $\mu = 0.8$ corresponds to the IP

Four-fermion model with $\mathcal{PT}$-type symmetry in $D = 2 + 1$

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$$S_{\text{mix}}[\bar\psi, \psi] = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2x \left\{ \bar\psi \left(\not{\partial} + \gamma_3 \mu \right) \psi - \left[\tfrac{\lambda_S}{2N} \left(\bar\psi \, \psi \right)^2 + \tfrac{\lambda_V}{2N} \left(\bar\psi \, \mathrm{i}\gamma_\nu \, \psi \right)^2 \right] \right\}$$

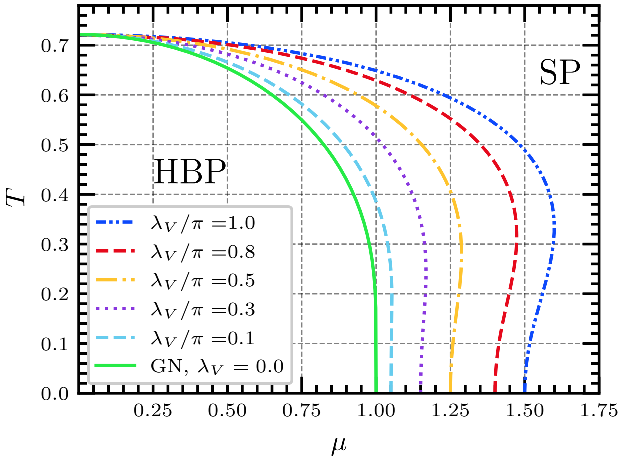
- Bosonization similar to before

$$S[\bar\psi, \psi, \sigma, \omega_\nu] = \int \mathrm{d}^3x \left[\bar\psi \, \left(\not{\partial} + \mathrm{i}\gamma_\nu \omega_\nu + \gamma_0 \mu + \sigma \right) \, \psi + \frac{\omega_\nu \omega_\nu}{2\lambda_V} + \frac{\sigma^2}{2\lambda_S} \right]$$

- Homogeneous condensation:
  - $\bar\omega_j = 0$ & $\omega_0 \sim \mathrm{i}\langle \psi^\dagger \psi \rangle / N$ purely imaginary; shift in chemical potential $\bar\mu = \mu + \mathrm{i}\bar\omega_0$
- Charge symmetry breaking at $\mu \neq 0$: $\mathcal{C}\omega_0 = -\omega_0$, but $\mathcal{CK}$ invariance – as in QCD!
- Complex saddle points $(\sigma, \omega_\nu) = \left(\bar\sigma, \mathrm{i}\bar n \delta_{0,\nu} \right)$!
- The following results are generic for all models with local $\left(\bar\psi \Gamma \psi \right)^2$ in $D = 2 + 1$

[MW, PRD **110**, 034008 (2024)]

$D = 2 + 1$ four-fermion model: Homogeneous phase boundary

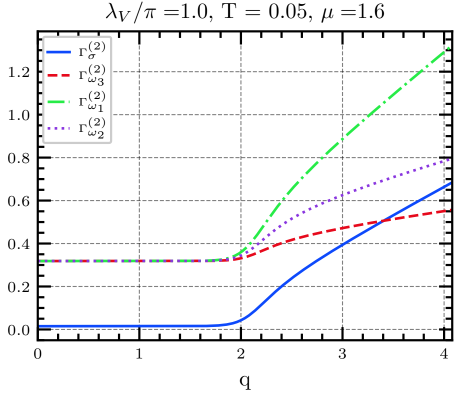


In general: Broken phase enlarged by vector coupling

Stability analysis in four-fermion model with $\mathcal{PT}$ symmetry

Mean-field stability analysis: Diagonalize Hessian $H_{\phi_j \phi_k}(q)$

- Symmetric phase: No IP & no moat regime
- Exponentially decaying propagators, "normal" symmetric phase



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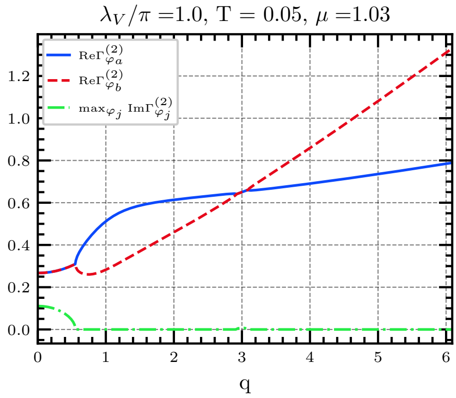
- Low momentum expansion of inverse propagators
- Poles are $q^2 = -\Gamma^{(2)}_{\phi}(q=0)\left(\frac{\mathrm{d}^2\Gamma^{(2)}_{\phi}}{\mathrm{d}q^2}\Big|_{q=0}\right)^{-1} \in \mathbb{C}$

$$\Rightarrow \langle\phi(x)\phi(0)\rangle \sim \mathrm{e}^{-mx}\sin(px)$$



Only σ and ω_0 studied

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- Low momentum expansion of inverse propagators
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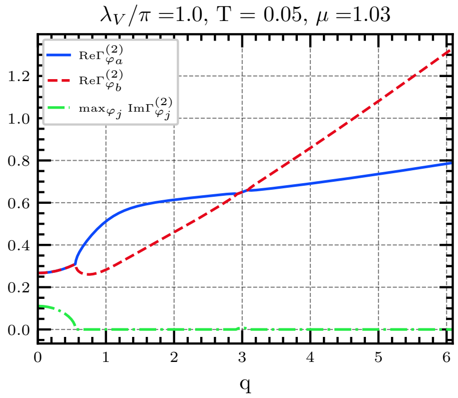
$$\Rightarrow \langle\phi(x)\phi(0)\rangle \sim \mathrm{e}^{-mx}\sin(px)$$

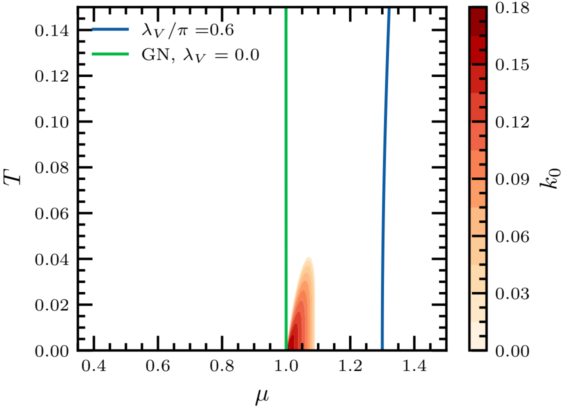
Quantum pion liquid, $\mathcal{PT}$ broken!

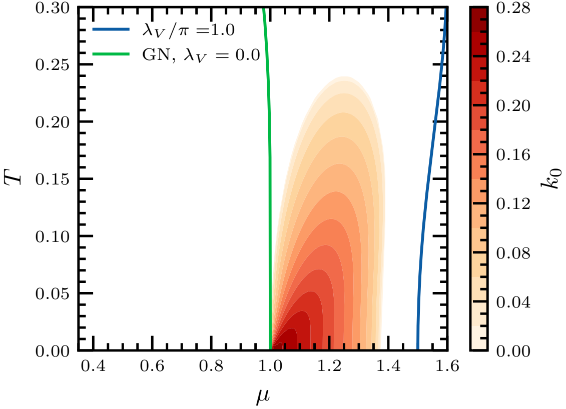


Only σ and ω_0 studied

Static Hessian: Complex-conjugate eigenvalues

$k_0 = \max_j \mathrm{Im}\Gamma^{(2)}_{\phi_j}(0)$ – scale for oscillation

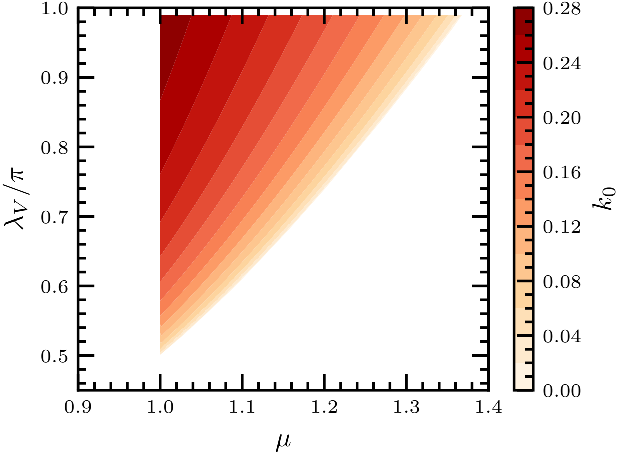




Analog of CEP at $d = 2$ spatial dimensions at $T = 0$

Static Hessian at zero temperature

Smooth onset of k_0 when increasing λ_V at fixed μ



Summary

Observation of quantum pion liquid (QπL)

- Spatially oscillating correlators $C(x) \sim \mathrm{e}^{-mx} \sin(kx)$ in NJL-type models $(D = 2 + 1)$

  [MW, PRD **110**, 034008 (2024)]

  - Related to $\mathcal{CK}$ invariance of FF model
  - Mixing between scalar and vector mesons; Competing attractive and repulsive interactions
  - Similar effects reported in Polyakov-Loop quark-meson model with ω

    [Haensch, Rennecke, von Smekal, PRD **110**, 036018(2024)0, arXiv:2308.16244]

  - QπL relevant around region of CEP

- Regimes with spatially modulations very relevant in QCD at $\mu \neq 0$ ($\mathcal{PT}$ symmetry!)
  - All mechanisms discussed above also apply to QCD at $\mu \neq 0$

Scalar $O(N)$ model: Influence of bosonic fluctuations on inhomogeneous condensates

Motivation and main idea

Motivation: Study the influence of bosonic fluctuation on an IP in a simple and effective setup

- Assuming IPs exist in some approximation, how do the bosonic fluctuations impact the inhomogeneous condensate ?

Main idea

- Quark determinant, bosonic loops, etc., can be expanded in powers of mesonic fields $\vec{\phi}, \partial\vec{\phi}$ in an effective theory for 'chiral physics' (see, e.g., Quark-Meson or Nambu-Jona-Lasinio model)
- At intermediate density and temperature CSB dominated by light-mesons $\vec{\phi} = (\sigma, \vec{\pi}) \Rightarrow \mathrm{O}(4)$ model for two-flavor QCD

$$\langle\sigma\rangle \sim \langle\bar{\psi}\psi\rangle \quad \langle\vec{\pi}\rangle \sim \langle\bar{\psi}\mathrm{i}\vec{\tau}\gamma_5\psi\rangle$$

$\Rightarrow$ Study $\mathrm{O}(N)$ model as effective theory

Studying IPs in an $\mathrm{O}(N)$ model

$$\mathcal{L} = \frac{1}{2}\left(\partial_0\vec{\phi}\right)^2 + \frac{1}{2}\left(\partial_j\vec{\phi}\right)^2 + \frac{m^2}{2}\vec{\phi}^2 + \frac{\lambda N}{4}(\vec{\phi}^2)^2$$

Studying IPs in an $O(N)$ model

CRC-TR 211

$$\mathcal{L} = \frac{1}{2}\left(\partial_0 \vec{\phi}\right)^2 + \frac{Z}{2}\left(\partial_j \vec{\phi}\right)^2 + \frac{m^2}{2}\vec{\phi}^2 + \frac{\lambda N}{4}(\vec{\phi}^2)^2$$

- Finite density breaks Lorentz-invariance $\Rightarrow Z \neq 1$

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$$\mathcal{L}_{\mathrm{HO}} = \frac{1}{2M^2}\left(\sum_j \partial_j^2\vec{\phi}\right)^2$$

- Resulting theory is super-renormalizable, bare propagators $\sim \mathbf{p}^{-4}$

$\mathrm{O}(N)$ model in the large-N limit

CRC-TR 211

$$L_{\text{eff}} = \frac{Z}{2}\left(\partial_j \vec{\phi}\right)^2 + \frac{1}{2M^2}\left(\sum_j \partial_j^2 \vec{\phi}\right)^2 + \frac{m^2}{2}\vec{\phi}^2 + \frac{\lambda N}{4}(\vec{\phi}^2)^2$$

- Large-N limit easily solvable with constraint field approach, λN & M is fixed

  [Pisarski, Tsvelik, Valgushev, PRD $\mathbf{102}, 016015\ (2020)$] [Moshe, Zinn-Justin, Phys. Rept. $\mathbf{385}, 69-228\ (2003)$]

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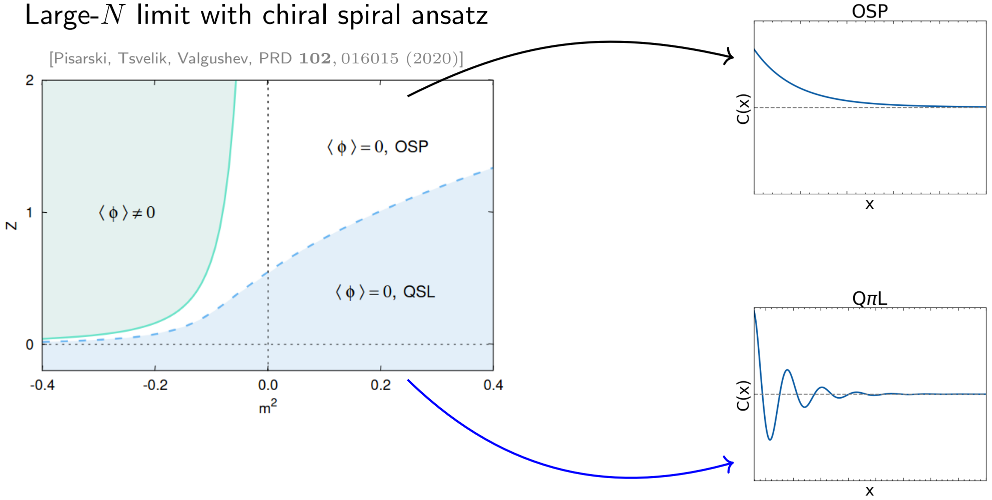
- As expected: $k_0 \neq 0$ is solution for classical EoM

Large-N limit with chiral spiral ansatz

[Pisarski, Tsvelik, Valgushev, PRD **102**, 016015 (2020)]



$$\mathrm{Q}\pi\mathrm{L}: \langle\phi^i(x)\phi^j(0)\rangle|_{x\to\infty} \sim \delta^{ij}\mathrm{e}^{-m_r} c_1 \cos(m_i x)!$$

Beyond large-N with lattice field theory

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  - "Phase diagram" in (Z, m^2) plane and dependence on N

Observables

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Simulation setup

- So far: lattice volumes $V = 12^3, 16^3, 20^3$, results rather stable in this range
- Thermalization: $\sim 10^2$ Metropolis steps, ~ 1000 HMC steps
- Statistics: $4000 - 10000$ configurations used
- Jackknife error: Typically smaller than dots

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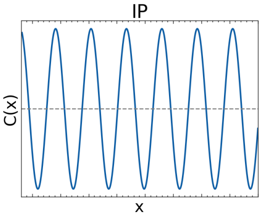
Distinguish via fits to spatial correlator C^{ij}

IP

- Spatial mesonic correlators $C(x) = \langle \phi(x)\phi(y=0)\rangle$ are oscillatory functions $C_{\rm osc}$ at large x

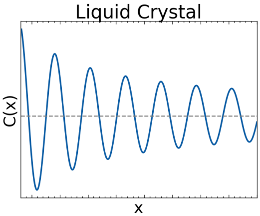
$\langle \phi_j \rangle \sim \langle \bar\psi \Gamma_j \psi \rangle = f(\mathbf{x})$



Liquid crystal

- Disordering through Goldstone modes of translational symmetry breaking (phonons)

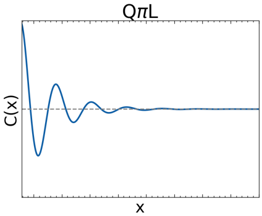
$$C(x) \sim |x|^{-\beta} C_{\rm osc}(x)$$



Quantum spin / π liquid

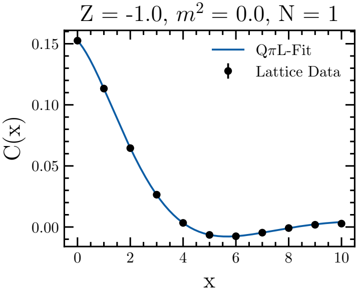
- Disordering through $\mathrm{O}(N-2)$ goldstone modes

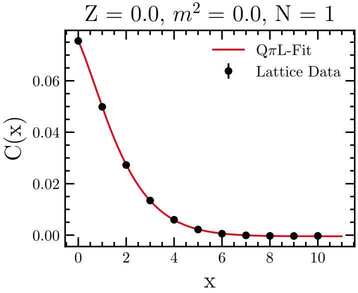
$$C(x) \sim \mathrm{e}^{-mx} C_{\rm osc}(x)$$



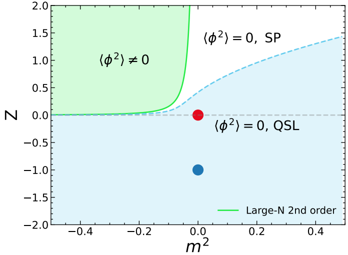
Correlators $N = 1$

Correlators are (discrete) rotationally symmetric, study $C(\mathbf{x}) = C((x, 0, 0))$





Large-N



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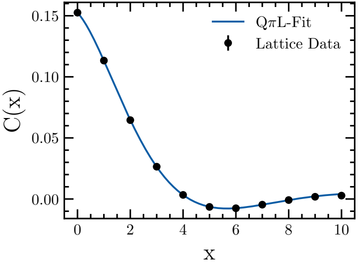

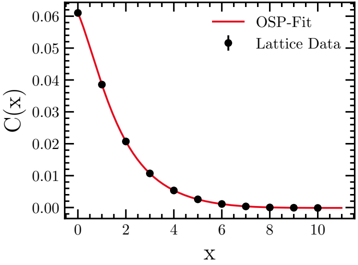

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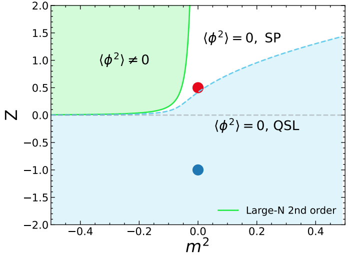

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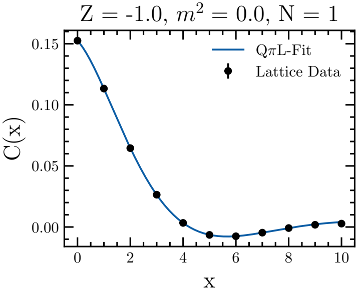

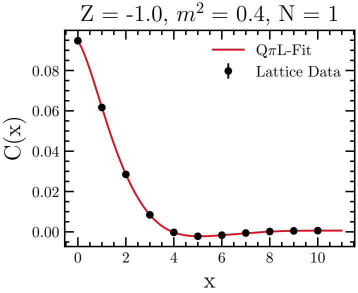

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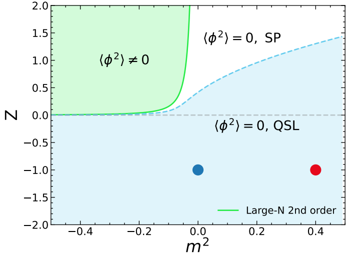

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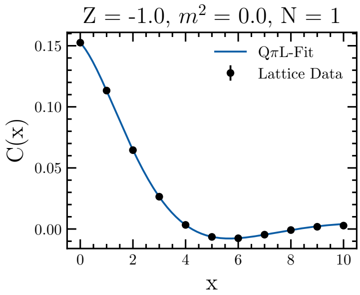

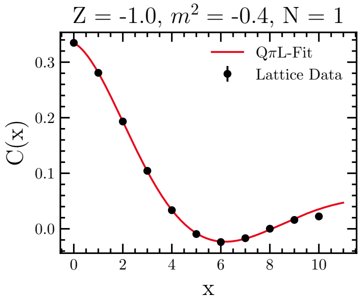

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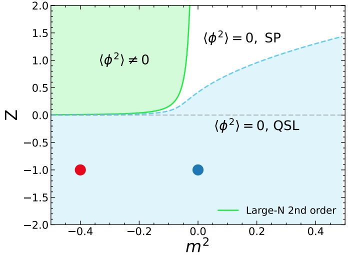

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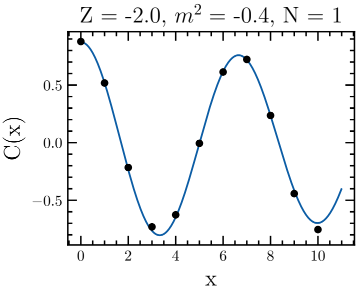

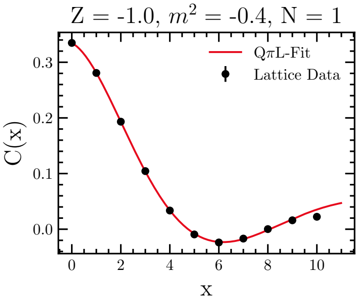

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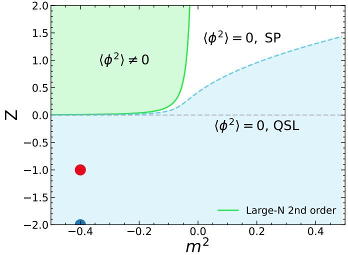
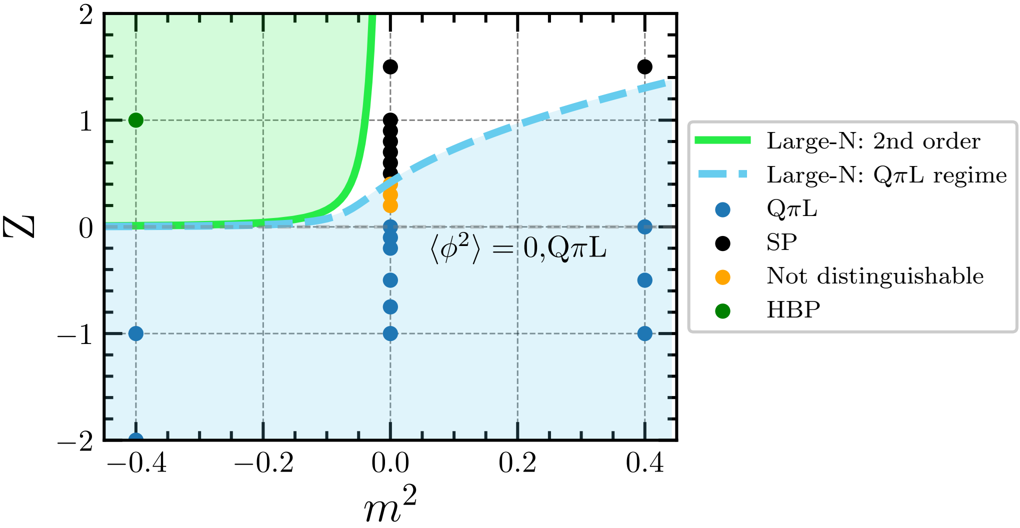
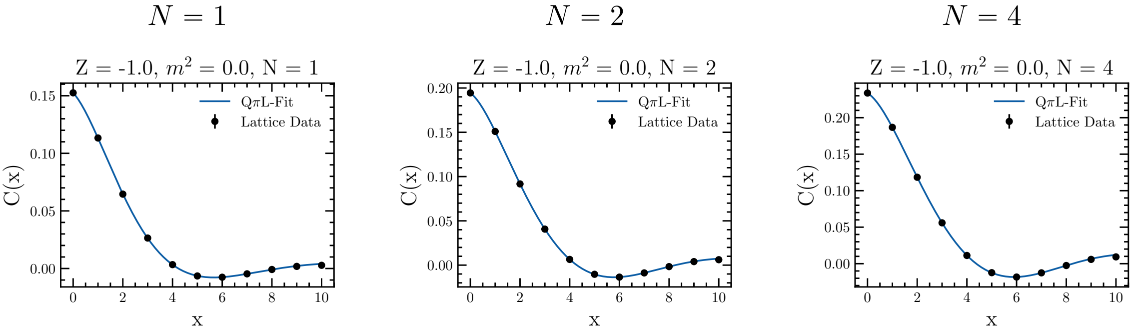


'Phase diagram' $N = 1$



Independence of N



- $m^2 = 0.0, Z \in [-1.0, 1.0]$: Phase diagram almost independent of $N (= 1, 2, 4, 8, 10)$
- Mechanism through disordering with transverse modes might not be the full picture

Summary

- Bosonic quantum fluctuations lead to disordering of inhomogeneous condensate
- Again: $\mathrm{Q}\pi\mathrm{L}$ with oscillating (damped) correlation functions
- Regime almost independent of N & also $N = 1, 2$
  $\Rightarrow$ Mechanism of disordering through $N-2$ Goldstone bosons cannot be the full explanation
- Important constraint: Finite lattice spacing and, especially, finite volume

Open ends

- Infinite volume extrapolation using inhomogeneous external field $h(x)$ for $N = 1, 2, 4$ to clarify fate of IP finally
- (Discrete) rotational symmetry might even be present with external field
- The fate of the $O(N)$ symmetry with the external field: Study $N = 2, 4$

Experimental signatures and conclusions

Experimental signatures

- Observable signatures for spatial modulated phases are hard to grasp:
  - Quantum fluctuations play an important role
  - The knowledge is far from complete, especially in QCD
- A precursor of these regimes is the so-called moat regime with $Z < 0$ for some meson field, e.g., pions

$$\mathcal{L}_{\text{eff}} = \frac{Z}{2}\left(\partial_j \vec{\phi}\right)^2 + \frac{1}{2M^2}\left(\sum_j \partial_j^2 \vec{\phi}\right)^2 + \frac{m^2}{2}\vec{\phi}^2 + \frac{\lambda N}{4}(\vec{\phi}^2)^2$$

- Experimental consequences of moats and inhomogeneities have been studied in the past years, e.g., peaks in two-particle correlation functions
  - Rennecke, Pisarski, Rischke, PRD $\mathbf{107}, 116013$ (2023), arXiv: 2301.11484
  - Pisarski, Rennecke, PRL $\mathbf{127}, 152302$ (2021), arXiv: 2103.06890
  - Fukushima, Hidaka, Inoue, Shigaki, Yamaguchi, PRC $\mathbf{109}, 051903$ (2024)
- What other signatures do exist?

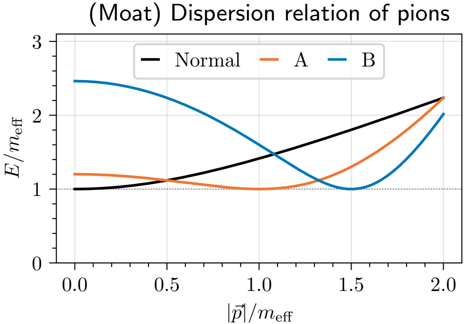
Dilepton production in moat regime

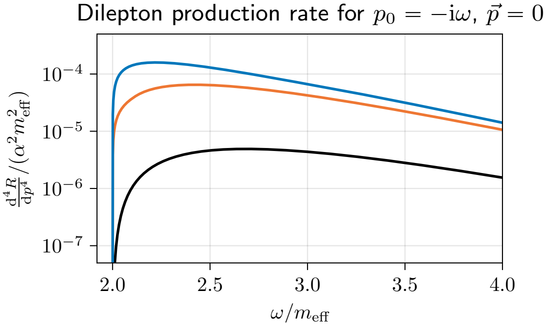
- Dilepton production via photon self polarization Π [McLerran, Toimela, PRD $\mathbf{31}, 545$ (1985)]

$$\frac{d^4R}{dp^4}(P) = \frac{\alpha}{12\pi^4}\,\frac{n(\omega)}{-P^2}\,\mathrm{Im}\,\Pi^{\mu\mu}(p_0,\vec{p})\ .$$

- Consider Dileptons with vanishing spatial momenta, i.e., $P = (p_0, 0)$, analytic continuation $p_0 = -\mathrm{i}\omega$
- Gauging $\mathcal{L}_{\mathrm{eff}} \Rightarrow$ Contribution to Π from "moaty pions"
- $\partial_\mu \to D_\mu = \partial_\mu - \mathrm{i}eA_\mu$ in $\mathcal{L}_{\mathrm{eff}}$ is not enough
- 8 independent operators of mass dimensions 6 that lead to a coupling of A_μ and ϕ
- For $\vec{p} = 0$: Only $Z(\partial_j\vec{\phi})^2$ and $(\partial_j^2\vec{\phi})^2/M^2$ contribute to $\mathrm{Im}\Pi$

[Nussinov, Ogilvie, Pannullo, Pisarski, Rennecke, Schindler, MW, arXiv: 2410.22418]





$\Rightarrow$ Enhancement of dilepton production an important signature for moat regime (and other spatially modulated regimes?)

Signals for $\vec{p} \neq 0$ under investigation

Conclusions

- Suggested in this talk: Quantum pion liquid: Oscillatory two-point correlation functions, homogeneous one-point functions, no translational symmetry breaking
  - Obtained in two model approaches with different focus and field content
    1. Fermionic theory: $\mathrm{Q}\pi\mathrm{L}$ generated by mixing effects between scalar and vector mesons
    2. Scalar $O(N)$ model as theory of light mesons: $\mathrm{Q}\pi\mathrm{L}$ seems to survive in full simulation, IP does not! (for $N = 4$, while for $N = 1$ unclear)
- Phases with oscillatory behavior are highly relevant for QCD at finite density
  - Existence of $\mathrm{Q}\pi\mathrm{L}$ & IP aligns with expectations from the emergent $\mathcal{CK}$ symmetry. This symmetry is also present in QCD at $\mu \neq 0$
  - Inhomogeneous chiral condensates might be disordered through fluctuations
  - Robust: Moat dispersion relation that favors oscillatory behavior in mesonic observables
- What are experimental consequences of this oscillatory behavior ?
  - Two particle correlations, Dilepton production rate. . .

Appendix

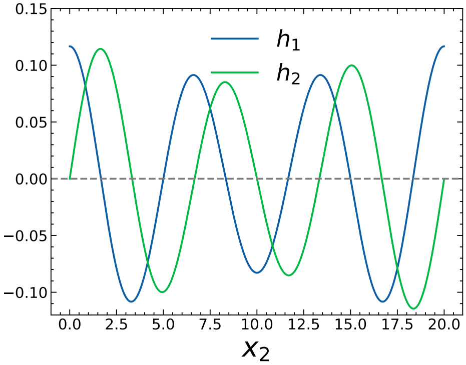
External breaking of translational & rotational symmetry

$$L_{h_0} = L_{\text{eff}} - \vec{h}(x)\vec{\phi}(x)$$

$$h(x) = \frac{h_0}{\sqrt{2\pi}L\,\sigma}\sum_{n=0}^{L-1} \mathrm{e}^{-\frac{1}{2\sigma}(p_n-k_0)^2}\begin{pmatrix}\cos(p_n x_2)\\ \sin(p_x x_2)\\ \vec{0}\end{pmatrix}$$

- Idea: External breaking through parameter h_0 and $h_0 \to 0$ to ensure that there is no symmetry breaking
- Extract momentum parameter k_0 through $h_0 = 0$ fit
- $\langle \phi^i(x)\phi^j(y)\rangle$ loose translational invariance in x_2 direction and rotational symmetry

$$C(x,y) = C(x_0 - y_0, x_1 - y_1, x_2, y_2)$$



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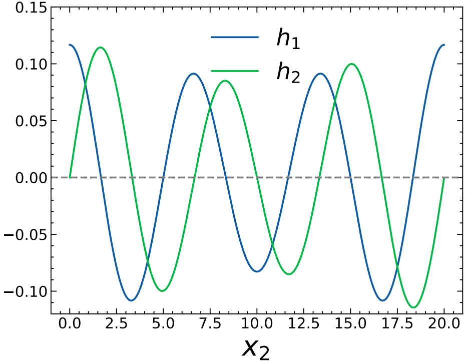

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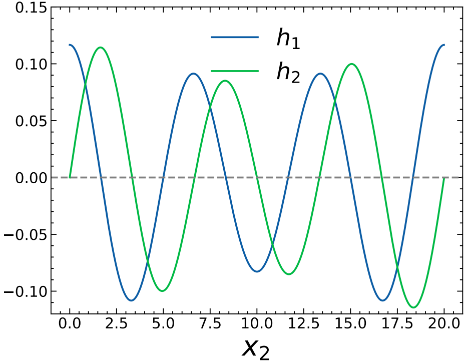
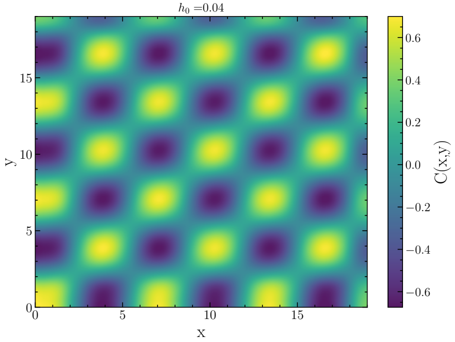
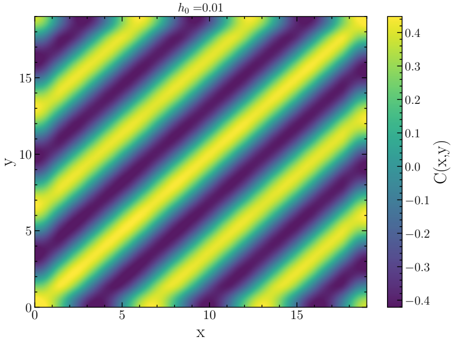


External breaking for $N = 1$

$C(0,0,x,y)$ for $Z = -1.5, m^2 = 0.0$

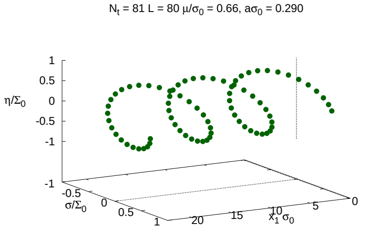


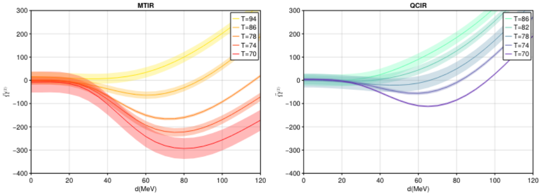


Translational symmetry clearly broken for $|h_0| > 0.01$
Symmetry $C(x,y) = f(x-y)$ is restored at $h_0 = 0.01, -0.01$

$\Rightarrow$ Compute $h_{0,c}(V)$ to determine fate of IP vs $Q\pi L$!

Exotic phases at $\mu \neq 0$ from functional methods: DSE





- QCD Dyson-Schwinger setup with gluon propagator fitted to quenched lattice data
- Perturbative quark-loop effects

$\Rightarrow$ Chiral density wave is self-consistent solution of DSE

[Müller, Buballa, Wambach, PL B **727**, 240 (2013)]

- Stability analysis of 2PI effective action in rainbow ladder approximation
- Below certain temperature: $\langle\bar{\psi}\psi\rangle = 0$ is unstable with respect to IP
- Analysis can only be trusted on left spinodal where $\langle\bar{\psi}\psi\rangle = \text{const.} \neq 0$

[Motta et al., arXiv:2406.00205 (2024)]

$D = 2+1$ four-fermion model: General stability analysis

$$S[\bar\psi,\psi,\vec\phi] = \int \mathrm{d}^3x \left\{ N \sum_j \frac{\phi_j^2}{2\lambda} + \bar\psi Q \psi \right\}, \qquad Q = \not{\partial} + \gamma_0\mu + \sum_j c_j\,\phi_j,$$

- Curvature is diagonalizable

$$\Gamma^{(2)}_{\phi_j}\left(M^2,\mu,T,q^2\right) = \frac{1}{\lambda} - \ell_1\left(M^2,\mu,T\right) + L_{2,\phi_j}(M^2,\mu,T,q^2)$$

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CRC-TR 211

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- Momentum dependence fully contained in $L_{2,\phi_j}(M^2, \mu, T, q^2)$

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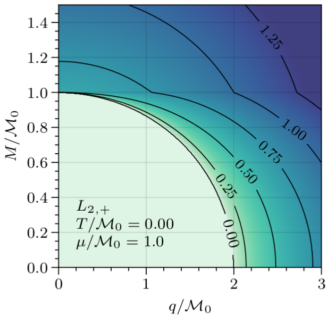
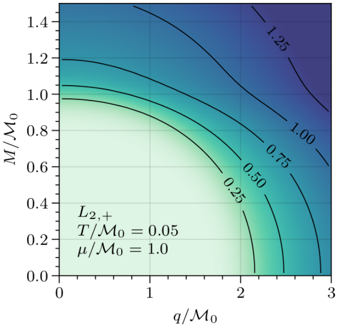
  - $L_{2,\phi_j} = L_{2,-} = -\mathbf{q}^2\ell_2(\mathbf{q}^2)$ – also monotonically increasing – no instabilities
- Remember: $\Gamma^{(2)}(q \neq 0) < 0$ necessary for IP

Momentum dependence of two point function $L_{2,+}/L_{2,-}$

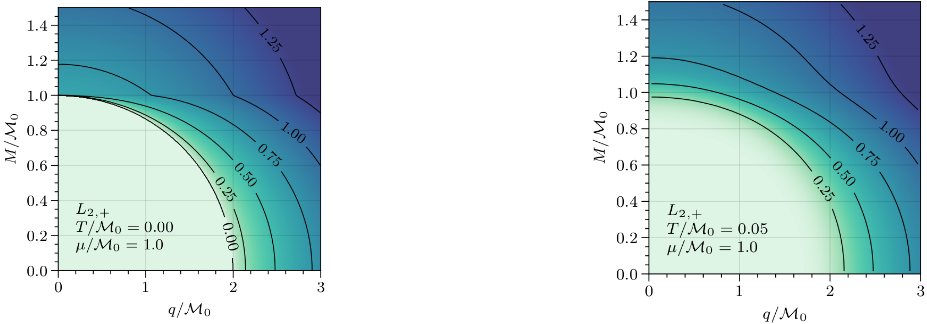




- $L_{2,+}/L_{2,-}$ are monotonically increasing functions of q for all M

[Pannullo, MW, PRD **108**, 036011 (2023) arXiv:2305.09444]

Momentum dependence of two point function $L_{2,+}/L_{2,-}$



- $L_{2,+}/L_{2,-}$ are monotonically increasing functions of q for all M

No IP and no moat regime in all models containing (some of) these 16 interaction channels

$$\bar{\psi} c_j \phi_j \psi$$

$$(c_j)_{j=1,\ldots,16} = (1, \mathrm{i}\gamma_4, \mathrm{i}\gamma_5, \gamma_{45}, \vec{\tau}, \mathrm{i}\vec{\tau}\gamma_4, \mathrm{i}\vec{\tau}\gamma_5, \vec{\tau}\gamma_{45})$$

[Pannullo, MW, PRD **108**, 036011 (2023) arXiv:2305.09444]

Examples (I): FF and Yukawa models

Model	Used channels c_j	Field basis $\vec{\varphi}_j$ diagonalizing $\mathcal{S}^{(2)}_{\mathrm{eff}}$	Momentum dependence of $\Gamma^{(2)}_{\varphi_j}$		Symmetry groups
			$L_{2,+}$	$L_{2,-}$	
GN	1	σ	σ		$\mathrm{U}_{\mathbb{I}_4}(N) \times \mathrm{U}_{\gamma_{45}}(N) \times \mathbb{Z}_{\gamma_5}(2) \times \mathrm{SU}_{\vec{\tau}}(2) \times P_4 \times P_5$
NJL	$1, \mathrm{i}\vec{\tau}\gamma_4, \mathrm{i}\vec{\tau}\gamma_5$	$\sigma, \vec{\pi}_4, \vec{\pi}_5$	σ	$\vec{\pi}_4, \vec{\pi}_5$	$\mathrm{U}_{\mathbb{I}_4}(N) \times \mathrm{U}_{\gamma_{45}}(N) \times \mathrm{SU}_{A,\gamma_4}(2N) \times \mathrm{SU}_{A,\gamma_5}(2N) \times \mathrm{SU}_{\vec{\tau}}(2) \times P_4 \times P_5$
$\chi\mathrm{HGN}_P$	$1, \mathrm{i}\gamma_4, \mathrm{i}\gamma_5, \gamma_{45}$	$\sigma, \eta_4, \eta_5, \eta_{45}$ (for $\bar{\eta}_{45} = 0$)	σ, η_{45}	η_4, η_5	$\mathrm{U}_{\gamma}(2N) \times \mathrm{SU}_{\vec{\tau}}(2) \times P_4 \times P_5$
PSFF	$1, \mathrm{i}\gamma_4, \mathrm{i}\gamma_5, \gamma_{45}, \vec{\tau}, \mathrm{i}\vec{\tau}\gamma_4, \mathrm{i}\vec{\tau}\gamma_5, \mathrm{i}\vec{\tau}\gamma_{45}$	$\sigma, \eta_4, \eta_5, \eta_{45}, \vec{a}_0, \vec{\pi}_4, \vec{\pi}_5, \vec{\pi}_{45}$ (for $\bar{\eta}_{45} = \bar{\vec{\pi}}_{45} = 0$)	$\sigma, \eta_{45}, \vec{\varsigma}, \vec{\pi}_{45}$	$\eta_4, \eta_5, \vec{\pi}_4, \vec{\pi}_5$	$\mathrm{U}_{\gamma}(2N) \times \mathrm{SU}_{\vec{\tau}}(2) \times P_4 \times P_5$

IPs and moat regime absent also for Yukawa versions of these models

[Pannullo, MW, PRD **108**, 036011 (2023) arXiv:2305.09444]

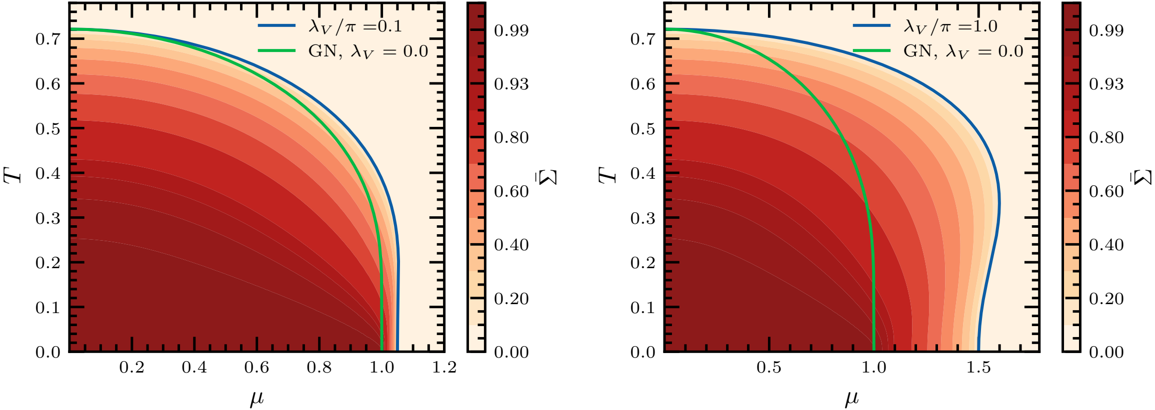
Examples (II): Multiple chemical potentials

Used channels c_j	Bosonic auxiliary fields ϕ_j	Non-zero chemical potentials	Field basis $\vec{\varphi}_j$ diagonalizing $\mathcal{S}^{(2)}_{\mathrm{eff}}$	Momentum dependence of $\Gamma^{(2)}_{\varphi_j}$ $f(M^2,\mu) = L_{2,+}(M^2,\mu,T,\mathbf{q}^2)$	Underlying symmetry group
$1, \gamma_{45}$	σ, η_{45}	$\mu_L = \mu + \mu_{45}$ $\mu_R = \mu - \mu_{45}$	$\phi_L = (\sigma + \eta_{45})$ $\phi_R = (\sigma - \eta_{45})$	$f(\bar{\phi}_L^2, \mu_L)$ $f(\bar{\phi}_R^2, \mu_R)$	$\mathrm{U}_{\mathbb{I}_4}(N) \times \mathrm{U}_{\gamma_{45}}(N) \times$ $\mathbb{Z}_{\gamma_5}(2) \times \mathrm{SU}_{\vec{\tau}}(2) \times$ $P_4 \times P_5$
$1, \tau_3$	$\sigma, a_{0,3}$	$\mu_\uparrow = \mu + \mu_I$ $\mu_\downarrow = \mu - \mu_I$	$\phi_\uparrow = (\sigma + \varsigma_3)$ $\phi_\downarrow = (\sigma - \varsigma_3)$	$f(\bar{\phi}_\uparrow^2, \mu_\uparrow)$ $f(\bar{\phi}_\downarrow^2, \mu_\downarrow)$	$\mathrm{U}_{\mathbb{I}_4}(N) \times \mathrm{U}_{\gamma_{45}}(N) \times$ $\mathbb{Z}_{\gamma_5}(2) \times \mathrm{U}_{\tau_3}(1) \times$ $P_4 \times P_5$
$1, \tau_3\gamma_{45}$	$\sigma, \pi_{45,3}$	$\mu_{L,\uparrow} = \mu_L + \mu_I$ $\mu_{L,\downarrow} = \mu_L - \mu_I$ $\mu_{R,\uparrow} = \mu_R + \mu_I$ $\mu_{R,\downarrow} = \mu_R - \mu_I$	$\varphi_+ = (\sigma + \pi_{45,3})$ $\varphi_- = (\sigma - \pi_{45,3})$	$f(\bar{\varphi}_+^2, \mu_{L,\uparrow})$ $+f(\bar{\varphi}_+^2, \mu_{R,\downarrow})$ $f(\bar{\varphi}_-^2, \mu_{L,\downarrow})$ $+f(\bar{\varphi}_-^2, \mu_{R,\uparrow})$	$\mathrm{U}_{\mathbb{I}_4}(N) \times \mathrm{U}_{\gamma_{45}}(N) \times$ $\mathbb{Z}_{\gamma_5}(2) \times \mathrm{U}_{\tau_3}(1) \times$ $P_4 \times P_5$
$1, \tau_3, \gamma_{45}$	$\sigma, a_{0,3}, \eta_{45}$	$\mu_{L,\uparrow}, \mu_{L,\downarrow},$ $\mu_{R,\uparrow}, \mu_{R,\downarrow}$	$\phi_L = (\sigma + \eta_{45})$ $\phi_R = (\sigma - \eta_{45})$ $a_{0,3}$	$f((\bar{\phi}_L + \bar{a}_{0,3})^2, \mu_{L,\uparrow})$ $+f(\bar{\phi}_L - \bar{a}_{0,3})^2, \mu_{L,\downarrow})$ $f((\bar{\phi}_R + \bar{a}_{0,3})^2, \mu_{R,\uparrow})$ $+f(\bar{\phi}_R - \bar{a}_{0,3})^2, \mu_{R,\downarrow})$ $\Gamma^{(2)}_{\phi_L} + \Gamma^{(2)}_{\phi_R}$	$\mathrm{U}_{\mathbb{I}_4}(N) \times \mathrm{U}_{\gamma_{45}}(N) \times$ $\mathbb{Z}_{\gamma_5}(2) \times \mathrm{U}_{\tau_3}(1) \times$ $P_4 \times P_5$

With multiple chemical potentials the analysis is straightforward only for a limited number of interaction channels

[Pannullo, MW, PRD **108**, 036011 (2023) arXiv:2305.09444]

Homogeneous phase diagram: Chiral condensate



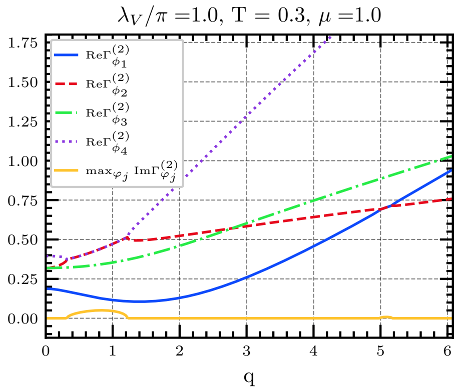
Now to the interesting object $H_{\phi_j \phi_k}$!

Mixing effect $H_{\sigma\omega_0} \sim \bar{\Sigma} \Rightarrow$ New physics in broken phase!

[MW, PRD **110**, 034008 (2024)]

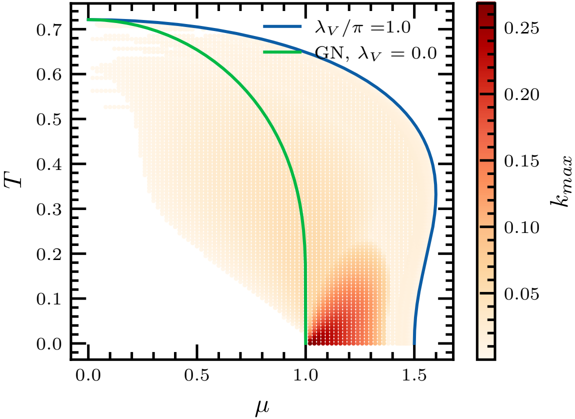
Non-vanishing imaginary part at $q \neq 0$

- At larger T: $H(0)$ has real eigenvalues, but $H(q \neq 0)$ develops compl. conj. eigenvalue pairs
- Caused by mixing of spatial vector components ω_j
- Interpretation of this phenomenon unclear so far

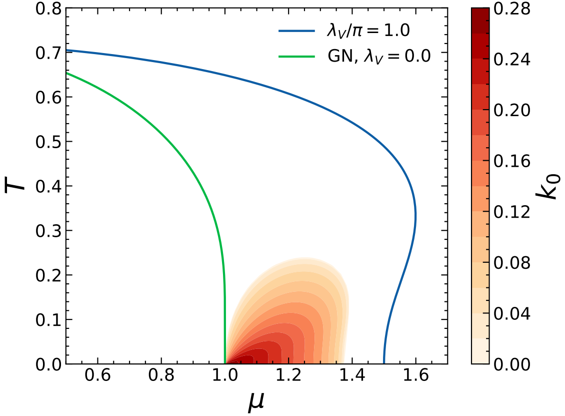


Complex-conjugate eigenvalues for $q \neq 0$

$$k_{\max} = \max_{j,q}\left(\mathrm{Im}\Gamma^{(2)}_{\phi_j}(q)\right)$$

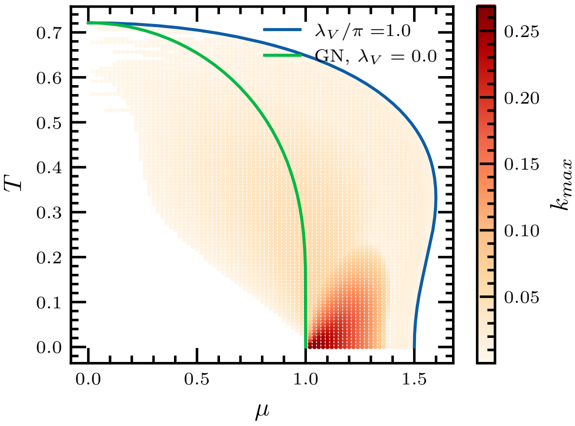


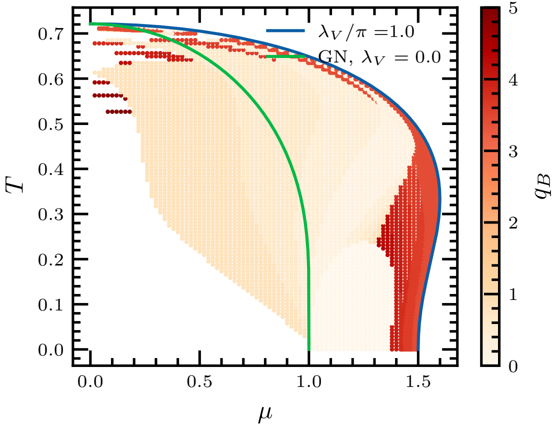
Static case – $k_0 = \max_j \mathrm{Im}\Gamma^{(2)}_{\phi_j}(0)$



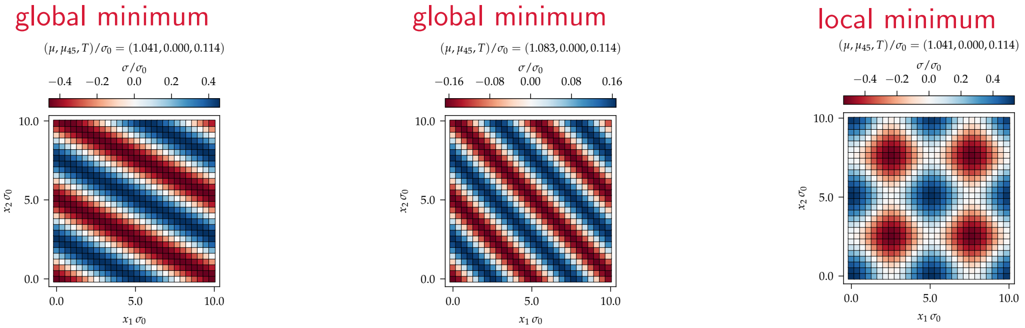
Complex-conjugate eigenvalues for $q \neq 0$

$$k_{\max} = \max_{j,q}\left(\mathrm{Im}\Gamma^{(2)}_{\phi_j}(q)\right) \qquad q_B = \min_{q\in C}, C = \{q \in [0,\infty) | \mathrm{Im}\Gamma^{(2)}_{\phi_j}(q) \neq 0\}$$





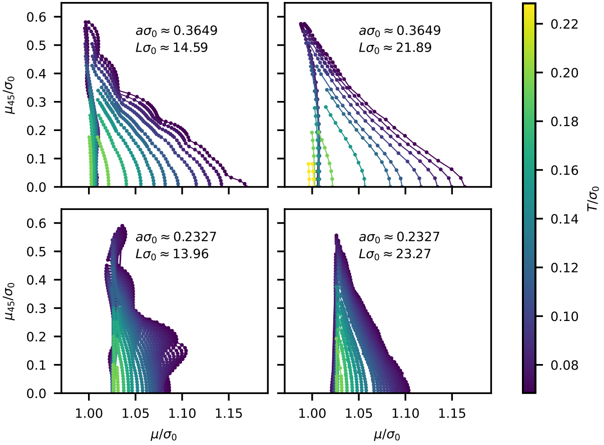
- Lattice field theory: Stability analysis & brute for minimization
- At finite lattice spacing: inhomogeneous groundstates
- Thus: Do we also find an inhomogeneous phase in the continuum?



[Buballa, Kurth, Wagner, MW, PRD **103**, 034503 (2020), arXiv: 2012.09588], [Pannullo, Wagner, MW, Symmetry **14**, 265 (2022), arXiv:2112.11183]

Chiral/Isospin imbalance in the Gross-Neveu model

Stability analysis on the lattice of GN model with μ_{45} or μ_I



[Pannullo, Wagner, MW, Symmetry **14**, 265 (2022), arXiv:2112.11183]