# ROBUST PORTFOLIO DECISIONS WITH HIGHER ORDER MOMENTS

## DECOMPOSITION SCHEME FOR SPARSE POLYNOMIAL OPTIMISATION PROBLEMS

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# Outline

1. Introduction
2. Portfolio Decisions
3. Decomposition Scheme for Sparse POPs
4. Discussion

# Outline

# General Overview

## Two nonconvex applications in finance

- Mean - Variance - Skewness - Kurtosis Portfolio Selection
- Robust Counterpart with Discrete Uncertainty Sets

## Two decomposition algorithms

- Partitioning Procedure for (dense) POPs
- Decomposition-based method for sparse POPs

# MOTIVATION

- Classical Mean-Variance Optimisation (MVO) (Markowitz)
- Asset returns are assumed normally distributed
- Incorporation of (central) moments higher than variance
- Resulting model is a Polynomial Optimisation Problem:

## POLYNOMIAL OPTIMISATION PROBLEM (POP)

$$\begin{aligned} p^* = \quad & \min_{x \in X} \quad p(x) \\ & \text{s.t.} \quad g_i(x,y) \geqslant 0 \quad i = 1, \ldots, m \end{aligned}$$

- POPs are Global Optimisation Problems
- Recent advances in Global Optimisation of POPs (Lasserre, Parrilo, Waki et al. )
- Ongoing research on decomposition-based methods for POPs

# Motivation

- Classical Mean-Variance Optimisation (MVO) (Markowitz)
- Asset returns are not normally distributed
- Incorporation of (central) moments higher than variance
- Resulting model is a Polynomial Optimisation Problem:

## Polynomial Optimisation Problem (POP)

$$p^* = \min_{x \in X} \; p(x) \quad \text{s.t.} \quad g_i(x,y) \geqslant 0 \quad i = 1, \ldots, m$$

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# Mean & Variance

- $R_{it}$ denotes the return on asset $i$ at time $t$
- $N$ the total number of returns on asset $i$
- $R_i$ random variable representing the average return on asset $i$

## Mean

- First order moment:

$$\mu_i = E[R_i] = \frac{1}{N}\sum_{t=1}^{N} R_{it}$$

- Expected or average value

## Variance

- Second order central moment:

$$\sigma_{ii} = E[(R_i - \mu_i)^2] = \frac{1}{N}\sum_{t=1}^{N} (R_{it} - \mu_i)^2$$

- Measure of dispersion around the mean

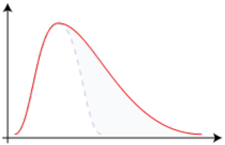
# Skewness

- Third order central moment:

$$s_{iii} = E[(R_i - \mu_i)^3] = \frac{1}{N}\sum_{t=1}^{N}(R_{it} - \mu_i)^3$$

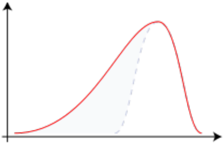
- Measure of asymmetry of the distribution
- Mean farther out in the long tail
- Long tail either to the right or to the left:

- *Positive* skewness
- *Right-skewed* distribution
- Few extreme gains



Positive Skew

- *Negative* skewness
- *Left-skewed* distribution
- Few extreme losses



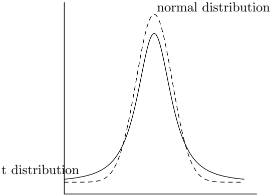
Negative Skew

# KURTOSIS

- Fourth order central moment:

$$k_{iiii} = E[(R_i - \mu_i)^4] = \frac{1}{N}\sum_{t=1}^{N}(R_{it} - \mu_i)^4$$

- Measure of peakedness of the distribution
- A distribution with *high* kurtosis is characterised by:
  - Symmetry
  - *Sharp* peak
  - *Fat* / *long* tails



# History

- Incorporation of Higher Moments dates back in early 60s (!)
- Substantial number of old-dated works investigated the persistence of asymmetries and/or fat tails in asset returns:
  - Mandelbrot (1963) and Fama (1965)
  - Simkowitz et al. (1978)
  - Singleton et al. (1986)
  - ...
- More recent works included skewness and/or kurtosis in portfolio selection:
  - Lai (1991) and Chunhachinda et al. (1997)
  - Athayde et al. (2004) and Harvey (2004)
  - Jondeau et al. (2006)
  - Mencia et al. (2009)
- Investor's preferences:
  - Scott et al. (1980) showed that investors generally like odd moments and dislike the even ones

# Objectives / Contributions

- Formulate the Mean - Variance - Skewness - Kurtosis portfolio optimisation problem (MVSKO)
- Handle the MVSKO in a (general) global optimisation framework
- Introduce, for the first time, the **robust** counterpart of MVSKO
- For discrete uncertainty sets, tackle the robust MVSKO in a global optimisation framework

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# Asset Statistics (Moments/Co-moments)





- Consider a portfolio of holdings in $n$ assets
- Marginal moments are not enough to describe the multivariate distribution
- Need to consider the co-moments as well
- Moments and co-moments constitute the *asset statistics*

| ***Asset Statistics*** | ***Symbols*** | ***Expressions*** |
|---|---|---|
| ***Mean*** $i$ | $\mu_i$ | $E[R_i]$ |
| ***Co-Var*** $i, j$ | $\sigma_{ij}$ | $E[(R_i - \mu_i)(R_j - \mu_j)]$ |
| ***Co-Skew*** $i, j, k$ | $s_{ijk}$ | $E[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)]$ |
| ***Co-Kurt*** $i, j, k, l$ | $k_{ijkl}$ | $E[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)(R_l - \mu_l)]$ |

# MVSKO

- Vector of portfolio weights: $x \in \mathbb{R}^n$
- Asset statistics assumed exact

| | *Asset Statistics* | *Portfolio Moments* | *Concise Notation* |
|---|---|---|---|
| ***M*** | $\mu = [\mu_i] \in \mathbb{R}^n$ | $\sum_{i=1}^{n} \mu_i x_i$ | $\mu^T x$ |
| ***V*** | $\Sigma = [\sigma_{ij}] \in \mathbb{R}^{n \times n}$ | $\sum_{i,j=1}^{n} \sigma_{ij} x_i x_j$ | $x^T \Sigma x$ |
| ***S*** | $S = [s_{ijk}] \in \mathbb{R}^{n \times n^2}$ | $\sum_{i,j,k=1}^{n} s_{ijk} x_i x_j x_k$ | $x^T S (x \otimes x)$ |
| ***K*** | $K = [k_{ijkl}] \in \mathbb{R}^{n \times n^3}$ | $\sum_{i,j,k,l=1}^{n} k_{ijkl} x_i x_j x_k x_l$ | $x^T K (x \otimes x \otimes x)$ |

- Budget & no short selling constraints:

$$X = \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1, x_i \geqslant 0\}$$

- *Kronecker product* for $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times q}$:

$$A \otimes B = \begin{bmatrix} a_{11} B & \dots & a_{1n} B \\ \dots & & \\ a_{m1} B & \dots & a_{mn} B \end{bmatrix} \in \mathbb{R}^{mp \times nq}$$

# MVSKO (CONT.)

## CLASSICAL MVO

- For input parameters $\lambda_1 + \lambda_2 = 1, \lambda_1, \lambda_2 \geqslant 0$:

$$\max_{x \in X} \lambda_1 \mu^T x - \lambda_2 x^T \Sigma x$$

## MVSKO

- For input parameters $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \lambda_1, \ldots, \lambda_4 \geqslant 0$:

$$\max_{x \in X} \lambda_1 \mu^T x - \lambda_2 x^T \Sigma x + \lambda_3 x^T S(x \otimes x) - \lambda_4 x^T K(x \otimes x \otimes x)$$

- The MVSKO is POP of total degree 4

# Robust MVSKO

- Vector of portfolio weights: $x \in \mathbb{R}^n$
- Asset statistics not exact
- Discrete uncertainty sets $\mathcal{U}$.

| | *Asset Statistics* | *Portfolio Moments* |
|---|---|---|
| ***M*** | $\mu \in \mathcal{U}_\mu$ | $\min_{\mu \in \mathcal{U}_\mu} \mu^T x$ |
| ***V*** | $\Sigma \in \mathcal{U}_\Sigma$ | $\max_{\Sigma \in \mathcal{U}_\Sigma} x^T \Sigma x$ |
| ***S*** | $S \in \mathcal{U}_S$ | $\min_{S \in \mathcal{U}_S} x^T S(x \otimes x)$ |
| ***K*** | $K \in \mathcal{U}_K$ | $\max_{K \in \mathcal{U}_K} x^T K(x \otimes x \otimes x)$ |

- Budget & no short selling constraints:

$$X = \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1, x_i \geqslant 0\}$$

# ROBUST MVSKO (CONT.)



- Our model for discrete uncertainty sets is:

$$\max_{x \in X} \min_{\substack{\mu \in \mathcal{U}_\mu, \Sigma \in \mathcal{U}_\Sigma \\ S \in \mathcal{U}_S, K \in \mathcal{U}_K}} \lambda_1 \mu^T x - \lambda_2 x^T \Sigma x + \lambda_3 x^T S (x \otimes x) - \lambda_4 x^T K (x \otimes x \otimes x)$$

- or:

$$\max_{\substack{x \in X \\ z_1, z_2, z_3, z_4}} \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 + \lambda_4 z_4$$

$$\text{s.t.} \quad \begin{array}{rcll} \mu^{(k_1)T} x & \geqslant & z_1 & k_1 = 1, \ldots, |\mathcal{U}_\mu| \\ -x^T \Sigma^{(k_2)} x & \geqslant & z_2 & k_2 = 1, \ldots, |\mathcal{U}_\Sigma| \\ x^T S^{(k_3)} (x \otimes x) & \geqslant & z_3 & k_3 = 1, \ldots, |\mathcal{U}_S| \\ -x^T K^{(k_4)} (x \otimes x \otimes x) & \geqslant & z_4 & k_4 = 1, \ldots, |\mathcal{U}_K| \end{array}$$

- The robust MVSKO is (sparse) POP of total degree 4

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# Motivation & Contribution

## Underlying Theory

- Benders Decomposition (BD) (Benders62)
- Sparse POPs (Lasserre06, Waki06)
- SDP (Alizadeh95)
  - Duality Theory (DT)
  - Extended Farkas Lemma (EFL)

## Basic Idea

- Extension of BD to SDP using DT and EFL
- Handle sparse POPs via their sparse SDP relaxations

# Semidefinite Programming (SDP)

## Primal SDP

$$
\begin{aligned}
z_1 = \quad & \min \quad c^T x \\
& \text{s.t.} \quad \mathcal{A}x = b \\
& \qquad\;\; x \succeq_{\mathcal{K}^n} 0
\end{aligned}
$$

## Dual SDP

$$
\begin{aligned}
z_2 = \quad & \max \quad b^T y \\
& \text{s.t.} \quad c - \mathcal{A}^T y \succeq_{\mathcal{K}^n} 0
\end{aligned}
$$

- $c,\ x \in R^{n^2}$, $\mathcal{A} \in R^{m \times n^2}$, $b,\ y \in R^m$
- $\mathcal{K}^n = \{x \in R^{n^2} \mid x = \mathbf{vec}(X); X \succeq 0\}$
- SDT, i.e. $z_1 = z_2$
- EFL, i.e. one of the two systems is consistent:

$$
c - \mathcal{A}^T y \succeq_{\mathcal{K}^n} 0 \tag{1}
$$

$$
u^T c = -1,\ u^T \mathcal{A} = 0,\ u \succeq_{\mathcal{K}^n} 0 \tag{2}
$$

- Solution of (2) is called the *Farkas dual solution*
- Well-known SDP solvers:
  - SeDuMi (*Matlab*)
  - CSDP & DSDP (*C*)
  - SDPA (*C++*)

# Convergent sparse SDP relaxations

- Consider the sparse POP:

$$p^* = \min_{x \in \mathbb{R}^n} \sum_{k=1}^{p} p_k(x_k)$$
$$\text{s.t.} \quad g_{j \in \mathcal{J}_k}(x_k) \geqslant 0 \quad k = 1, \ldots, p$$

  - $p$ disjoint sets $\mathcal{J}_k$: $\mathcal{J}_k \subset \{1, \ldots, m\}$ & $p$ sets $\mathcal{I}_k$: $\mathcal{I}_k \subset \{1, \ldots, n\}$
  - *cliques*: $\mathcal{I}_1, \ldots, \mathcal{I}_p$ & *coupling variables*: $\mathcal{I}_0' = \bigcap_{k=1}^{p} \mathcal{I}_k$

- Sparse SDP relaxation of order $\omega$:

$$p^*_\omega = \min_{y} \sum_{k=1}^{p} \sum_{\alpha_k \in N^n} p_{\alpha_k} y_{\alpha_k}$$
$$\text{s.t.} \quad M_\omega(y, \mathcal{I}_k) \succeq 0 \quad k = 1, \ldots, p$$
$$M_{\omega - d_j}(g_j y, \mathcal{I}_k) \succeq 0 \quad k = 1, \ldots, p$$
$$y_0 = 1$$

  - $2\omega \geqslant \max\{\deg f, \max_j \deg g_j\}$ & $\lim_{\omega \to \infty} p^*_\omega = p^*$ (Lasserre06)

- State of the art solver is **sparsePOP**:
  - *Matlab* solver (some *C*++ functions), SeDuMi for SDP problems

# Pre-process Phase (Partitioning of Variables)

- Sparsity pattern of POP is expressed by $\mathcal{I}_0', \mathcal{I}_1, \ldots, \mathcal{I}_p$
- SDP relaxation inherits sparsity pattern from POP (!)
- Partition moment variables according to SDP sparsity
- Moment variables correspond to *monomials*, i.e. products of powers of polynomial variables up to a certain degree

## Rule 1

The set of coupling *moment* variables derives from $\mathcal{I}_0'$

## Rule 2

The $i$-th set of independent *moment* variables derives from $\mathcal{I}_i$

# Master Problem Derivation

- Consider the sparse SDP problem:

$$
\begin{aligned}
p^*_\omega = \quad \min_{y, y^1, \dots, y^p} \quad & b^T y + \sum_{i=1}^{p} {d^i}^T y^i \\
\text{s.t.} \quad & T^i y + W^i y^i + h^i \succeq_{\mathcal{K}^m} 0 \quad i = 1, \dots, p \\
& Ay + c \succeq_{\mathcal{K}^v} 0
\end{aligned}
$$

- Fix coupling moment variables $y$
- Obtain $p > 1$ *subproblems*:

$$
\begin{aligned}
\rho_i(y) = \quad \min_{y^i} \quad & {d^i}^T y^i \\
\text{s.t.} \quad & W^i y^i + (h^i + T^i y) \succeq_{\mathcal{K}^m} 0
\end{aligned}
$$

- $b^T y + \sum_{i=1}^{p} \rho_i(y)$ is an upper bound on $p^*_\omega$

# Master Problem Derivation

- Consider the sparse SDP problem:

$$
\begin{aligned}
p_\omega^* = \min_{y, y^1, \ldots, y^p} \quad & b^T y + \sum_{i=1}^{p} {d^i}^T y^i \\
\text{s.t.} \quad & T^i y + W^i y^i + h^i \succeq_{\mathcal{K}^m} 0 \quad i = 1, \ldots, p \\
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\end{aligned}
$$

- $b^T y + \sum_{i=1}^{p} \rho_i(y)$ is an upper bound on $p_\omega^*$

# Projection

- The projected problem is:

$$\begin{aligned} p^*_\omega = \quad \min_y \quad & b^T y + \sum_{i=1}^{p} \rho_i(y) \\ \text{s.t.} \quad & Ay + c \succeq_{\mathcal{K}^v} 0 \\ & \rho_i(y) = \{ \inf_{y^i} {d^i}^T y^i \mid W^i y^i + (h^i + T^i y) \succeq_{\mathcal{K}^m} 0 \} \, \forall i \\ & y \in V \end{aligned}$$

- where

$$V = \{ y \mid W^i y^i + (h^i + T^i y) \succeq_{\mathcal{K}^m} 0 \text{ for some } y_i \ \forall i \}$$

## Employing DT:

$$\rho_i(y) = \{ \sup_{\lambda \succeq_{\mathcal{K}^m} 0} (-h^i - T^i y)^T \lambda^i \mid {W^i}^T \lambda^i = d^i \}$$

# Projection

- The projected problem is:

$$
\begin{aligned}
p_\omega^* = \quad \min_y \quad & b^T y + \sum_{i=1}^{p} \rho_i(y) \\
\text{s.t.} \quad & Ay + c \succeq_{\mathcal{K}^v} 0 \\
& \rho_i(y) = \{ \sup_{\lambda \succeq_{\mathcal{K}^m} 0} (-h^i - T^i y)^T \lambda^i \mid W^{i^T} \lambda^i = d^i \} \; \forall i \\
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& \qquad y \in V
\end{aligned}
$$

- where

$$V = \{ y \mid W^i y^i + (h^i + T^i y) \succeq_{\mathcal{K}^m} 0 \text{ for some } y_i \ \forall i \}$$

- Let:
  - $\Lambda^i = \{ {W^i}^T \lambda^i = d^i,\ \lambda^i \succeq_{\mathcal{K}^m} 0 \}$
  - $U^i = \{ {W^i}^T u^i = 0,\ u^i \succeq_{\mathcal{K}^m} 0 \}$

## Employing EFL:

$$y \in V \Leftrightarrow (h^i + T^i y)^T u^i \geqslant 0,\ \forall u^i \in U^i,\ i = 1, \dots, p$$

# PROJECTION

- The projected problem is:

$$\begin{aligned} p^*_\omega = \quad & \min_y \quad b^T y + \sum_{i=1}^{p} \rho_i(y) \\ & \text{s.t.} \quad Ay + c \succeq_{\mathcal{K}^v} 0 \\ & \qquad \rho_i(y) = \{ \sup_{\lambda \succeq_{\mathcal{K}^m} 0} (-h^i - T^i y)^T \lambda^i \mid {W^i}^T \lambda^i = d^i \} \, \forall i \\ & \qquad 0 \geqslant (-h^i - T^i y)^T u^i, \; \forall u^i \in U^i, \; i = 1, \ldots, p \end{aligned}$$

- Let:
  - $\Lambda^i = \{{W^i}^T \lambda^i = d^i, \; \lambda^i \succeq_{\mathcal{K}^m} 0\}$
  - $U^i = \{{W^i}^T u^i = 0, \; u^i \succeq_{\mathcal{K}^m} 0\}$

## EMPLOYING EFL:

$$y \in V \Leftrightarrow (h^i + T^i y)^T u^i \geqslant 0, \; \forall u^i \in U^i, \; i = 1, \ldots, p$$

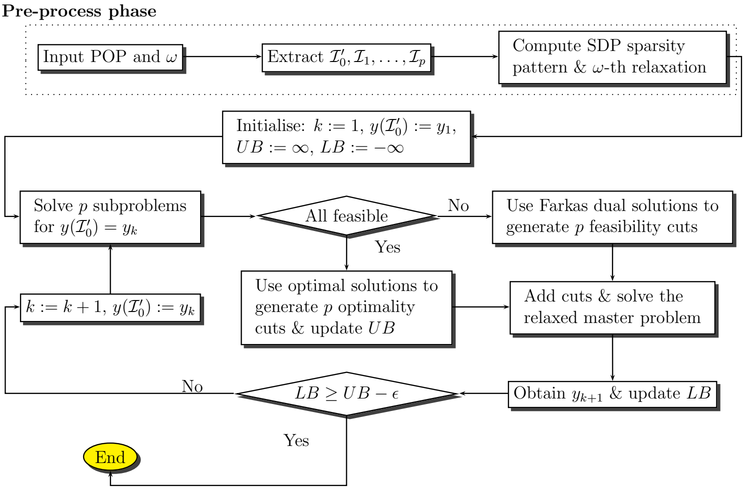
# Master Problem

- The projected problem is equivalent to:

$$
\begin{aligned}
p^*_\omega = \quad & \min_{y, z_1, \ldots, z_p} \quad b^T y + \sum_{i=1}^{p} z_i \\
& \text{s.t.} \quad Ay + c \succeq_{\mathcal{K}^\nu} 0 \\
& \qquad z_i \geqslant (-h^i - T^i y)^T \lambda^i, \ \forall \lambda^i \in \Lambda^i, \ i = 1, \ldots, p \\
& \qquad 0 \geqslant (-h^i - T^i y)^T u^i, \ \forall u^i \in U^i, \ i = 1, \ldots, p
\end{aligned}
$$

- Linear optimality/feasibility constraints
- Solve relaxed versions of the master problem
- Then, $b^T y + \sum_{i=1}^{p} z_i$ is a lower bound on $p^*_\omega$
- Obtain sequence of lower bounds
- Finite $\epsilon$-convergence

# Decomposition Scheme for Sparse POPs



# IMPLEMENTATION

- Implementation in C++
- Input files in *GAMS scalar* format
- Off the shelf functions for *Pre-Process Phase*:
  - CHOLMOD
  - SparsePOP
- Off the shelf functions for *Decomposition-based Method*:
  - CSDP
- C++ version of SparsePOP implemented, called *SparsePOP/CSDP*
- Compared results using metrics:

$$\epsilon_{p^*} = \frac{p^*_{\omega,bmrk} - p^*_{\omega}}{\max\{1, p^*_{\omega}\}}, \quad \epsilon_{x^*} = \max\left\{\frac{x^*_{i,bmrk} - x^*_i}{\max\{1, x^*_i\}}\right\},$$

- $x^*$, $p^*_\omega$ computed by our method
- $x^*_{bmrk}$, $p^*_{\omega,bmrk}$ computed by SparsePOP/CSDP
- $x^*_i$ ($x^*_{i,bmrk}$): $i$-th element of $x^*$ ($x^*_{bmrk}$)

# Benchmark Problems (GlobalLib)

- $\epsilon = 10^{-5}$, $d = 2$
- $0.01 \leqslant t \leqslant 2.75$ secs/iter & 0.37 secs/iter on average

| | | | | *SpPOP/CSDP* | *Decomposition-based Method* | | | |
|---|---|---|---|---|---|---|---|---|
| *Problem* | $n$ | $p^*$ | $\omega$ | $p^*_{\omega,bmrk}$ | $p^*_{\omega}$ | iters | $\epsilon_{p^*}$ | $\epsilon_{x^*}$ |
| **Bex2_1_2** | 6 | −213 | 1 | −214 | −214 | 3 | $\leqslant \epsilon$ | $\leqslant \epsilon$ |
| **Bex2_1_2** | 6 | −213 | 2 | −213 | −213 | 49 | $\leqslant \epsilon$ | $\leqslant \epsilon$ |
| **Bex9_1_1** | 13 | −13 | 1 | −13 | −13 | 10 | $\leqslant \epsilon$ | 0.01 |
| **Bex9_1_1** | 13 | −13 | 2 | −13 | −13 | 117 | $\leqslant \epsilon$ | 0.01 |
| **Bex9_1_5** | 13 | −1 | 1 | −1 | −1 | 2 | $\leqslant \epsilon$ | 0.4 |
| **Bex9_1_5** | 13 | −1 | 2 | −1 | −1 | 67 | $\leqslant \epsilon$ | 0.4 |
| **Bex9_2_8** | 6 | 1.5 | 1 | −78.5 | −78.5 | 1 | $\leqslant \epsilon$ | $\leqslant \epsilon$ |
| **Bex9_2_8** | 6 | 1.5 | 2 | 1.5 | 1.5 | 7 | $\leqslant \epsilon$ | $\leqslant \epsilon$ |
| **st_e21** | 6 | – | 1 | −14.1 | −14.1 | 4 | $\leqslant \epsilon$ | $\leqslant \epsilon$ |
| **st_e21** | 6 | – | 2 | −14.1 | −14.1 | 68 | $\leqslant \epsilon$ | $\leqslant \epsilon$ |
| **st_glmp_kk90** | 5 | – | 1 | −1758.51 | ⊠ | | | |
| **st_glmp_kk90** | 5 | – | 2 | 3 | 3 | 5 | $\leqslant \epsilon$ | $\leqslant \epsilon$ |

- ⊠ denotes unbounded problem

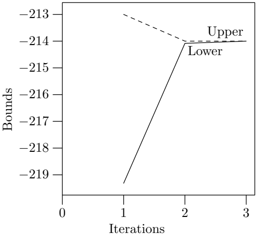
# Benchmark Problems (GlobalLib) (cont.)

- $\epsilon = 10^{-5}$, $d = 2$
- $0.01 \leqslant t \leqslant 2.75$ secs/iter & 0.37 secs/iter on average

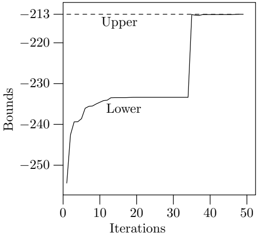
| | | | | *SpPOP/CSDP* | *Decomposition-based Method* | | | |
|---|---|---|---|---|---|---|---|---|
| *Problem* | $n$ | $p^*$ | $\omega$ | $p^*_{\omega,bmrk}$ | $p^*_{\omega}$ | iters | $\epsilon_{p^*}$ | $\epsilon_{x^*}$ |
| **Bex5_2_2_case1** | 9 | −400 | 1 | −574 | −574 | 16 | $\leqslant \epsilon$ | $\leqslant \epsilon$ |
| **Bex5_2_2_case2** | 9 | −600 | 1 | −1200 | −1200 | 15 | $\leqslant \epsilon$ | 0.02 |
| **Bex5_2_2_case3** | 9 | −750 | 1 | −825 | −825 | 19 | $\leqslant \epsilon$ | 0.1 |
| **Bex9_2_1** | 10 | 17 | 1 | 2 | 2 | 2 | $\leqslant \epsilon$ | $\leqslant \epsilon$ |
| **Bex9_2_1** | 10 | 17 | 2 | 11.53 | 11.53 | 354 | 0.0002 | 0.002 |
| **Bex9_2_2** | 10 | 99.9 | 1 | 99.9 | 99.9 | 4 | $\leqslant \epsilon$ | 0.001 |
| **Bex9_2_5** | 8 | 5 | 1 | 0 | 0 | 1 | $\leqslant \epsilon$ | $\leqslant \epsilon$ |
| **Bex9_2_6** | 16 | −1 | 1 | −1.5 | −1.5 | 23 | $\leqslant \epsilon$ | 0.001 |
| **Bex9_2_7** | 10 | 17 | 1 | 3.25 | 3.25 | 2 | $\leqslant \epsilon$ | $\leqslant \epsilon$ |
| **Bex9_2_7** | 10 | 17 | 2 | 12.45 | 12.44 | 298 | 0.0005 | 0.007 |
| **Bhaverly** | 12 | 900 | 1 | −600 | −600.1 | 8 | 0.0002 | 0.0004 |

# Convergent Bounds for Benchmark Problems Bex2_1_2, st_e21

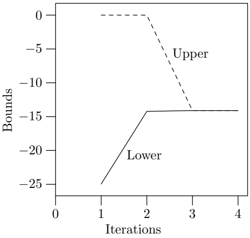
- $\omega = 1, p_1^* = -214$



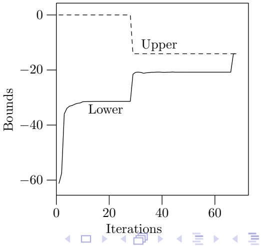

- $\omega = 2, p_2^* = -213$




- $\omega = 1, p_1^* = -14.1$




- $\omega = 2, p_2^* = -14.1$



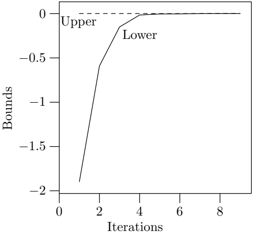
# ROBUST MVSKO

- $k = 4$ scenarios
- $\lambda_1 = \ldots = \lambda_4 = 0.25$
- $\epsilon = 10^{-5}$, $d = 4$, $\omega = 2$
- Some numerical difficulties

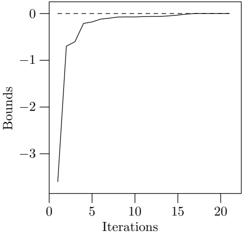
| $n$ | $\epsilon_{p^*}$ | $\epsilon_{x^*}$ | **iters** | **cputime/iter** | ***exited normally?*** |
|---|---|---|---|---|---|
| 2 | $\leqslant \epsilon$ | 0.002 | 9 | 0.6 | Y |
| 3 | $\leqslant \epsilon$ | 0.07 | 21 | 0.2 | Y |
| 4 | 0.002 | 0.9 | 26 | 0.6 | N |
| 5 | 0.03 | 0.9 | 500 | 15.01 | N |

# Convergent Bounds for Some Instances of Robust MVSKO

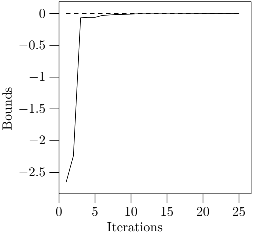
- $n = 2$ assets, $k = 4$ scenarios



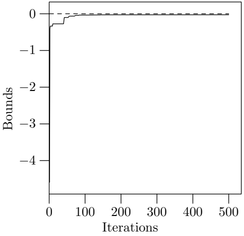

- $n = 3$ assets, $k = 4$ scenarios




- $n = 4$ assets, $k = 4$ scenarios




- $n = 5$ assets, $k = 4$ scenarios



# OUTLINE

# Conclusions & Future Work

- Extension of convex MVO to nonconvex MVSKO
- Extension of robust MVO to robust MVSKO
- General global optimisation framework
- Decomposition-based method for sparse POPs
  - BD extended to SDP
  - Sparse structure yields appropriate partitioning of variables
  - $p > 1$ independent subproblems in place of 1 in BD
  - Parallel implementation possible
  - Finite $\epsilon$-Convergence
  - Implementation in C++
  - Computational Experience in progress

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# Thank you for attending this talk

Questions?

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Comments?