Robust Portfolio Optimization with Derivative Insurance Guarantees

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Mean-Variance Portfolio Optimization

Optimal Asset Allocation Problem

Choose the weights vector $\boldsymbol{w} \in \mathbb{R}^n$ to make the portfolio return high, whilst keeping the associated risk $\rho(\boldsymbol{w})$ low.

- Mean-Variance Portfolio Optimization problem:

$$\begin{aligned} \underset{\boldsymbol{w} \in \mathbb{R}^n}{\text{maximize}} \quad & \boldsymbol{w}^T \boldsymbol{\mu} - \lambda \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w} \\ \text{subject to} \quad & \boldsymbol{w} \in \mathcal{W}. \end{aligned}$$

Solves tradeoff between expected return $\boldsymbol{w}^T \boldsymbol{\mu}$ and risk $\rho(\boldsymbol{w}) \equiv \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w}$, where λ is the risk aversion parameter.

Introduction to Robust Portfolio Optimization

- Let $\tilde{\boldsymbol{r}}$ denote the asset returns. Portfolio return is $\boldsymbol{w}^T\tilde{\boldsymbol{r}}$.
- Ben-Tal and Nemirovski [1], Rustem and Howe [3], suggest investor wants to maximize portfolio return:

$$\max_{\boldsymbol{w}\in\mathcal{W}} \ \boldsymbol{w}^T\tilde{\boldsymbol{r}}$$

- Assume that $\boldsymbol{r} \in \mathcal{U}$, where

$$\mathcal{U}_{\boldsymbol{r}} = \{\boldsymbol{r} \mid (\boldsymbol{r}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{r}-\boldsymbol{\mu}) \leq \delta^2\}$$

- Robust optimization takes worst-case approach:

$$\max_{\boldsymbol{w}\in\mathcal{W}} \min_{\boldsymbol{r}\in\mathcal{U}_{\boldsymbol{r}}} \ \boldsymbol{w}^T\boldsymbol{r} \equiv \max_{\boldsymbol{w}\in\mathcal{W}} \ \boldsymbol{w}^T\boldsymbol{\mu} - \delta\left\|\boldsymbol{\Sigma}^{1/2}\boldsymbol{w}\right\|_2.$$

Probabilistic Guarantees

- Assume we know the means $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma} \succ \mathbf{0}$ of the returns $\tilde{\boldsymbol{r}}$, but not the entire distribution.
- Let $\mathcal{P}$ be the set containing all the distributions that have mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$.
- El Ghaoui *et al.* [2] have shown for any $\boldsymbol{w} \in \mathcal{W}$

$$\delta = \sqrt{p/(1-p)} \implies \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{P}\{\boldsymbol{w}^T \tilde{\boldsymbol{r}} \geq \min_{\boldsymbol{r} \in \mathcal{U}_{\boldsymbol{r}}} \boldsymbol{w}^T \boldsymbol{r}\} = p$$

- Let $\phi^* = \max_{\boldsymbol{w} \in \mathcal{W}} \min_{\boldsymbol{r} \in \mathcal{U}_{\boldsymbol{r}}} \boldsymbol{w}^T \boldsymbol{r}$, then

$$\boldsymbol{w}^{*T} \boldsymbol{r} \geq \phi^* \quad \forall \boldsymbol{r} \in \mathcal{U}_{\boldsymbol{r}}.$$

The non-inferiority property of robust portfolios can be seen as a form of weak insurance.

Support Information and Coherency

- Assume we have support information about $\tilde{\boldsymbol{r}}$:

$$\mathcal{B} = \{\boldsymbol{r} : \boldsymbol{l} \leq \boldsymbol{r} \leq \boldsymbol{u}\} \quad \text{(Always true: } \mathcal{B} = \{\boldsymbol{r} : \boldsymbol{r} \geq \boldsymbol{0}\}\text{)}$$

- Can add support information to $\mathcal{U}_{\boldsymbol{r}}$:

$$\mathcal{U}_{\boldsymbol{r}} = \{\boldsymbol{r} \in \mathcal{B} \mid (\boldsymbol{r} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{r} - \boldsymbol{\mu}) \leq \delta^2\}$$

- By strong conic duality:

$$\max_{\boldsymbol{w} \in \mathcal{W}} \min_{\boldsymbol{r} \in \mathcal{U}_{\boldsymbol{r}}} \boldsymbol{w}^T \boldsymbol{r} \equiv \max_{\boldsymbol{w} \in \mathcal{W}, \boldsymbol{s} \geq \boldsymbol{0}} \boldsymbol{\mu}^T (\boldsymbol{w} - \boldsymbol{s}) - \delta \left\| \boldsymbol{\Sigma}^{1/2} (\boldsymbol{w} - \boldsymbol{s}) \right\|_2 .$$

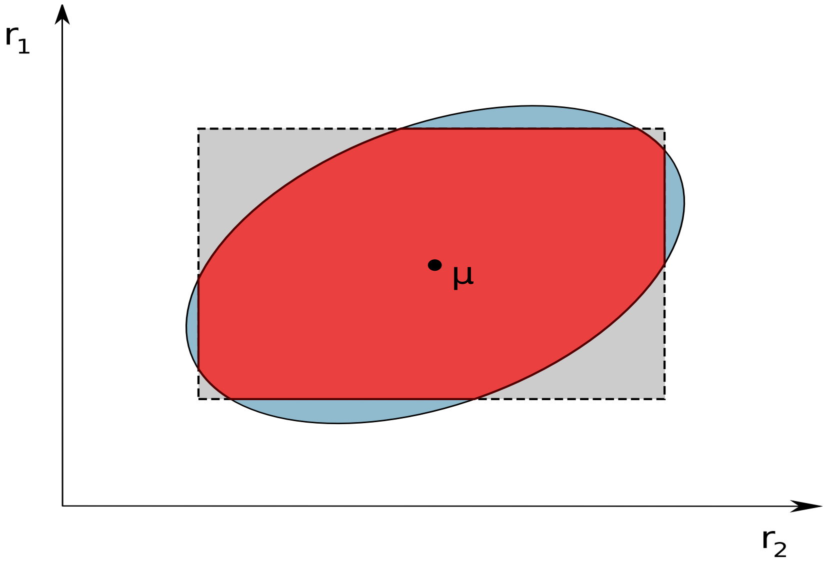
- Consider the function ρ:

$$\rho(\boldsymbol{w}) = \min_{\boldsymbol{s} \geq \boldsymbol{0}} -\boldsymbol{\mu}^T (\boldsymbol{w} - \boldsymbol{s}) + \delta \left\| \boldsymbol{\Sigma}^{1/2} (\boldsymbol{w} - \boldsymbol{s}) \right\|_2 .$$

  Can show that ρ is a coherent risk-measure.

- maximize worst-case return $\iff$ minimize coherent risk!

Uncertainty Set: Illustration



Parameter Uncertainty

- Have to estimate means $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma} \rightarrow$ considerable uncertainty.
- Portfolio optimization is very sensitive to errors in $\hat{\boldsymbol{\mu}} \rightarrow$ error-maximization effect.
- When $\tilde{\boldsymbol{r}}$ are i.i.d. then $\hat{\boldsymbol{\mu}}$ is approx. normally distributed:

$$\hat{\boldsymbol{\mu}} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Lambda}), \quad \boldsymbol{\Lambda} = (1/M)\boldsymbol{\Sigma}$$

- Can create uncertainty set for $\hat{\boldsymbol{\mu}}$

$$\mathcal{U}_{\boldsymbol{\mu}} = \{\boldsymbol{\mu} \mid (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}})^T \boldsymbol{\Lambda}^{-1} (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}) \leq \kappa^2, \ \boldsymbol{e}^T(\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}) = 0\}.$$

- Obtain uncertainty set for $\tilde{\boldsymbol{r}}$ that takes into account $\mathcal{U}_{\boldsymbol{\mu}}$:

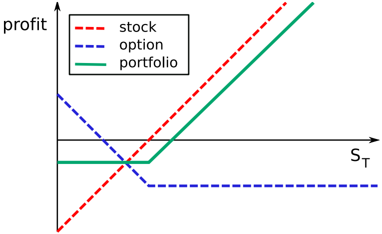
$$\mathcal{U}_{\boldsymbol{r}} = \{\boldsymbol{r} \in \mathcal{B} \mid \exists \boldsymbol{\mu} \in \mathcal{U}_{\boldsymbol{\mu}}, \ (\boldsymbol{r} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{r} - \boldsymbol{\mu}) \leq \delta^2\}$$

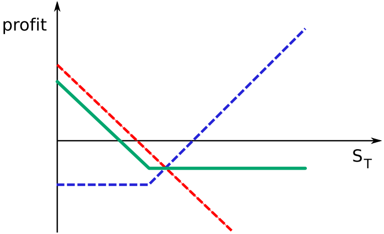
Insurance through Options

- Options allow us to limit the worst-case portfolio return.

$$V(S_T) = S_T + \max\{K - S_T, 0\} = \max\{S_T, K\}$$

$$V(S_T) = -S_T + \max\{S_T - K, 0\} = \max\{-S_T, -K\}$$





- Only consider European options maturing at investment horizon.

Modelling Option Returns

- Let $\boldsymbol{w^d}$ and $\tilde{\boldsymbol{r}}^{\boldsymbol{d}}$ denote option weights and returns resp.
- $\tilde{\boldsymbol{r}}$ uniquely determines $\tilde{\boldsymbol{r}}^{\boldsymbol{d}}$, thus $\tilde{\boldsymbol{r}}^{\boldsymbol{d}} \equiv f(\tilde{\boldsymbol{r}})$.
- Assume option j is a call with strike K_j and premium C_j on underlying stock i with initial price S_0^i, then $\tilde{r}_j^d$ is

$$\begin{aligned} f_j(\tilde{\boldsymbol{r}}) &= \frac{\max\left\{0, S_0^i \tilde{r}_i - K_j\right\}}{C_j} \\ &= \max\left\{0, a_j + b_j \tilde{r}_i\right\}, \text{ with } a_j = -\frac{K_j}{C_j} \text{ and } b_j = \frac{S_0^i}{C_j}. \end{aligned}$$

- Likewise, when option j is a put with premium P_j, then $\tilde{r}_j^d$ is

$$f_j(\tilde{\boldsymbol{r}}) = \max\left\{0, a_j + b_j \tilde{r}_i\right\}, \text{ with } a_j = \frac{K_j}{P_j} \text{ and } b_j = -\frac{S_0^i}{P_j}.$$

- In compact notation:

$$\tilde{\boldsymbol{r}}^{\boldsymbol{d}} = f(\tilde{\boldsymbol{r}}) = \max\left\{0, \boldsymbol{a} + \boldsymbol{B}\tilde{\boldsymbol{r}}\right\}.$$

Incorporating Options within Robust Framework

- Portfolio return is $\tilde{r}_p = \boldsymbol{w}^T \tilde{\boldsymbol{r}} + (\boldsymbol{w}^{\boldsymbol{d}})^T \tilde{\boldsymbol{r}}^{\boldsymbol{d}}$.
- We will always set $\boldsymbol{w}^{\boldsymbol{d}} \geq \boldsymbol{0}$, and $\boldsymbol{1}^T \boldsymbol{w} + \boldsymbol{1}^T \boldsymbol{w}^{\boldsymbol{d}} = 1$.
- Robust max-min problem formulation:

$$\max_{(\boldsymbol{w}, \boldsymbol{w}^{\boldsymbol{d}}) \in \mathcal{W}} \min_{\substack{\boldsymbol{r} \in \mathcal{U}_{\boldsymbol{r}}, \\ \boldsymbol{r}^{\boldsymbol{d}} = f(\boldsymbol{r})}} \boldsymbol{w}^T \boldsymbol{r} + (\boldsymbol{w}^{\boldsymbol{d}})^T \boldsymbol{r}^{\boldsymbol{d}}$$

- Equivalent semi-infinite problem formulation:

$$\begin{array}{ll} \underset{\boldsymbol{w}, \boldsymbol{w}^{\boldsymbol{d}}, \phi}{\text{maximize}} & \phi \\ \text{subject to} & \boldsymbol{w}^T \boldsymbol{r} + (\boldsymbol{w}^{\boldsymbol{d}})^T \boldsymbol{r}^{\boldsymbol{d}} \geq \phi \qquad \forall \boldsymbol{r} \in \mathcal{U}_{\boldsymbol{r}},\ \boldsymbol{r}^{\boldsymbol{d}} = f(\boldsymbol{r}) \\ & (\boldsymbol{w}, \boldsymbol{w}^{\boldsymbol{d}}) \in \mathcal{W} \end{array}$$

- At optimality ϕ^* is the worst-case portfolio return when $\boldsymbol{r} \in \mathcal{U}_{\boldsymbol{r}}$.

Incorporating Options within Robust Framework

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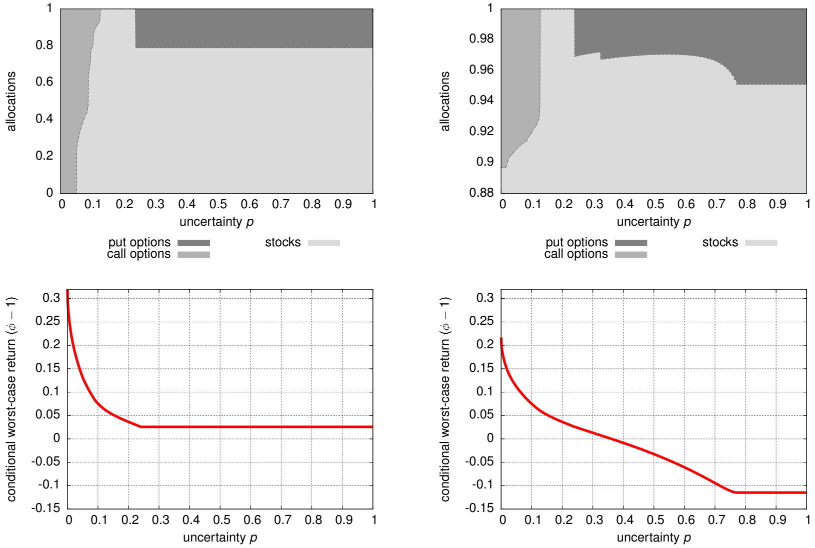
- Equivalent semi-infinite problem formulation:

$$\begin{aligned} \underset{\boldsymbol{w}, \boldsymbol{w^d}, \phi, \boldsymbol{y}, \boldsymbol{s}}{\text{maximize}} \quad & \phi \\ \text{subject to} \quad & \boldsymbol{\mu}^T (\boldsymbol{w} + \mathbf{B}^T \boldsymbol{y} - \boldsymbol{s}) - \delta \left\| \Sigma^{1/2} (\boldsymbol{w} + \mathbf{B}^T \boldsymbol{y} - \boldsymbol{s}) \right\|_2 + \boldsymbol{a}^T \boldsymbol{y} \geq \phi \\ & (\boldsymbol{w}, \boldsymbol{w^d}) \in \mathcal{W}, \ \mathbf{0} \leq \boldsymbol{y} \leq \boldsymbol{w^d}, \ \boldsymbol{s} \geq \mathbf{0} \end{aligned}$$

- At optimality ϕ^* is the worst-case portfolio return when $\boldsymbol{r} \in \mathcal{U}_{\boldsymbol{r}}$.

Worst-Case Return Behaviour



Insured Robust Portfolio Optimization

- At optimality we obtain the non-inferiority guarantee:

$$\boldsymbol{w}_*^T \boldsymbol{r} + (\boldsymbol{w}_*^{\boldsymbol{d}})^T \boldsymbol{r}^{\boldsymbol{d}} \geq \phi^* \qquad \forall \boldsymbol{r} \in \mathcal{U}_{\boldsymbol{r}},\ \boldsymbol{r}^{\boldsymbol{d}} = f(\boldsymbol{r})$$

- Extreme events can cause $\tilde{\boldsymbol{r}}$ to be realised outside $\mathcal{U}_{\boldsymbol{r}} \rightarrow$ no more guarantees!
- Control deterioration of portfolio return below ϕ for any realisation of $\tilde{\boldsymbol{r}}$:

$$\boldsymbol{w}^T \boldsymbol{r} + (\boldsymbol{w}^{\boldsymbol{d}})^T \boldsymbol{r}^{\boldsymbol{d}} \geq \theta\phi \qquad \forall \boldsymbol{r} \in \mathcal{B},\ \boldsymbol{r}^{\boldsymbol{d}} = f(\boldsymbol{r}),$$

  where $\theta \in [0, 1]$.
- Insurance guarantee expressed as fraction of ϕ:
  - Only hedge against extreme scenarios not covered by non-inferiority guarantee.
  - Prevents insurance from being overly expensive.

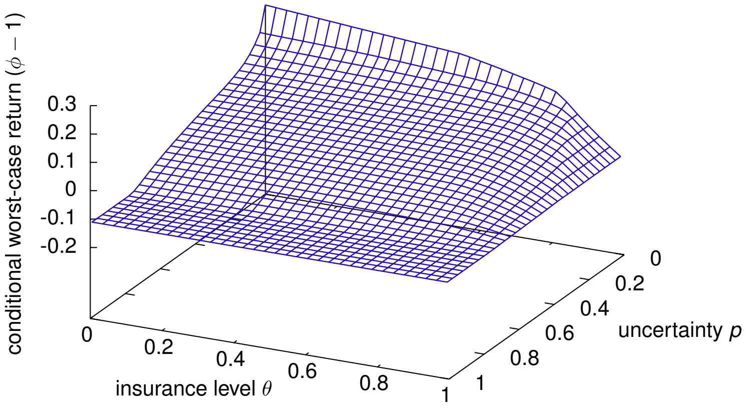
Guarantee Tradeoff

- The insured robust portfolio optimization model:

$$\begin{aligned} \underset{\boldsymbol{w},\boldsymbol{w^d},\phi}{\text{maximize}} \quad & \phi \\ \text{subject to} \quad & \boldsymbol{w}^T\boldsymbol{r} + (\boldsymbol{w^d})^T\boldsymbol{r^d} \geq \phi \qquad \forall \boldsymbol{r} \in \mathcal{U}_{\boldsymbol{r}},\ \boldsymbol{r^d} = f(\boldsymbol{r}) \\ & \boldsymbol{w}^T\boldsymbol{r} + (\boldsymbol{w^d})^T\boldsymbol{r^d} \geq \theta\phi \qquad \forall \boldsymbol{r} \in \mathcal{B},\ \boldsymbol{r^d} = f(\boldsymbol{r}) \\ & (\boldsymbol{w}, \boldsymbol{w^d}) \in \mathcal{W}. \end{aligned}$$

- Has a SOCP reformulation $\rightarrow$ tractable.
- Model exposes tradeoff between non-inferiority and insurance guarantees:
  - As $\mathcal{U}_{\boldsymbol{r}}$ increases, ϕ^* decreases.
  - When ϕ^* decreases, so does insurance level $\theta\phi^*$ and associated insurance costs (premium).

Guarantee Tradeoff



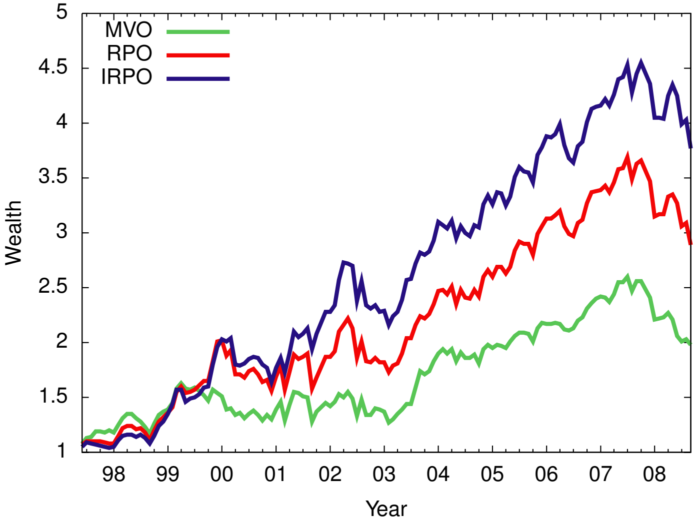
Backtest Results

- We consider the following indices in the portfolio:

Ticker	Name
XMI	AMEX Major Market Index
SPX	S&P 500 Index
MID	S&P Midcap 400 Index
SML	S&P Smallcap 600 Index
RUT	Russell 2000 Index
NDX	NASDAQ 100 Index

- Adopt a monthly rebalancing strategy.
- Each month, we include all the options on the indices maturing in one month (data by Optionmetrics).
- Calculate out-of-sample returns between 19/06/1997 and 10/10/2008.

Backtest Results



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