COMISEF Computational Optimization Methods in Statistics, Econometrics and Finance

- Marie Curie Research and Training Network funded by the EU Commission through MRTN-CT-2006-034270 -

COMISEF WORKING PAPERS SERIES

WPS-029 09/02/2010

Robust International Portfolio Management

R. J. Fonseca
W. Wiesemann
B. Rustem

Robust International Portfolio Management*

Raquel J. Fonseca†, Wolfram Wiesemann and Berç Rustem

Department of Computing
Imperial College of Science, Technology and Medicine
180 Queen's Gate, London SW7 2AZ, UK

February 9, 2010

Abstract

We present an international portfolio optimization model where we take into account the two different sources of return of an international asset: the local returns denominated in the local currency, and the returns on the foreign exchange rates. The explicit consideration of the returns on exchange rates introduces non-linearities in the model, both in the objective function (return maximization) and in the triangulation requirement of the foreign exchange rates. The uncertainty associated with both types of returns is incorporated directly in the model by the use of robust optimization techniques. We show that, by using appropriate assumptions regarding the formulation of the uncertainty sets, the proposed model has a semidefinite programming formulation and can be solved efficiently. While robust optimization provides a guaranteed minimum return inside the uncertainty set considered, we also discuss an extension of our formulation with additional guarantees through trading in quanto options for the foreign assets and in equity options for the domestic assets.

Keywords robust optimization, international portfolio optimization, quanto options, semidefinite programming

1 Introduction

The seminal work of Markowitz [20] in 1952 on portfolio optimization initiated great interest and further academic research in the area of risk management. It was only in 1968, however, that this same interest was extended to international portfolios, that is, to portfolios with assets denominated in foreign currency. In his seminal work, Grubel [13] suggests a model that explains how international capital movements are a function not only of the interest rate differential between countries, but also of the growth rate of asset holdings.

International portfolios are attractive from the point of view of risk diversification, as it is expected that assets in the same economy have a higher correlation among themselves than with assets in other countries. Levy and Sarnat [18]

*Financial support from the EU Commission through MRTN-CT-2006-034270 COMISEF is gratefully acknowledged.

†Corresponding author: rfonseca@imperial.ac.uk

present an estimate of the potential gains on international diversification for the period 1951-67. They conclude that the traditional approach of comparing the returns of the developed countries with those of the developing ones underestimated the impact of international diversification. The low correlation between these two different economies allows the reduction of the portfolio variance.

Eun and Resnick [9] question previous studies on the benefits of international diversification, as these did not take into account the uncertainty related to the estimation of the returns. Moreover, they alert for the risk associated with fluctuating exchange rates, as unfavorable movements have the potential to override asset gains. With this in mind, they propose a hedging strategy based on the short selling of an expected amount of foreign currency at the forward rate. They show that from the point of view of an US investor, this hedging strategy outperforms unhedged strategies.

The issue of hedging the currency risk has stimulated Black to suggest a universal hedging formula [4]. He suggests that foreign assets should be hedged equally by investors in all countries, though not fully. Glen and Jorion introduce forward contracts in order to protect against depreciations of the foreign exchange rates and optimize simultaneously over the weight allocation between assets and currencies [12]. They show that portfolios with hedging restrictions, limited to the size of the foreign asset holdings, in general perform better than portfolios with unitary or universal hedging strategies. Larsen and Resnick [17] note, however, that all published results relate to ex-post portfolios, and do not take into account the parameter uncertainty resulting from estimations. In their studies, they test the performance of hedging strategies when parameter inputs are estimated from historical data. They find that "when dealing with historical market data, the degree of parameter uncertainty is so severe that one cannot reliably base parameter inputs or sophisticated hedging strategies upon it." A survey of the topic may be found in Shawky *et al.* [24].

More recently, Topaloglou *et al.* [27, 28] present a multi-stage stochastic programming model that jointly determines the asset weights and the corresponding hedge ratios for the international currencies, using the Conditional Value-at-Risk (CVaR) as a risk measure. In their work, the authors include not only forward contracts but also currency and quanto options to hedge against the foreign exchange risk. They find that only "in-the-money" put options have a comparable performance with forward contracts. Because currency options are more flexible than forwards, it has been thought that these would be more appropriate hedging instruments, as the investor is not sure of the future cash inflow [11]. Steil, on the other hand, argues that the underlying of the currency option is the foreign exchange, which does not correspond to the contingency underlying the exposure, the asset return [25]. Instead, he suggests that quanto options, which convert the price of an underlying asset into another asset at a fixed guaranteed rate, are more adequate for international asset allocation problems [14].

We deal with the uncertainty inherent to parameter estimation in international portfolio optimization by applying robust optimization techniques. This idea has initially been developed by Rustem and Howe [23]. We expand on their work by reformulating the problem in a convex tractable framework and by subsequently implementing the model using historical market data. The paradigm of robust optimization gained the attention of the academic community after the simultaneous works of Ben-Tal and Nemirovski [3] and El-Ghaoui and Lebret

[8]. In the robust optimization framework, uncertainty is incorporated directly in the model by considering the returns as random variables. An uncertainty set is then specified, which reflects the investor's expectations of the future returns, and the model is optimized over this entire set. We refer the reader to Ben-Tal *et al.* [1] for a recent survey of robust optimization and its applications.

Thi study extends the scope of the work in currency only portfolios in Fonseca *et al.* [10] to the more complex case of a portfolio with assets priced in foreign currency. The main contributions of our work may be summarized as follows:

1. Application of robust optimization techniques to an international portfolio optimization problem where assets are denominated in different currencies.
2. Development of a semidefinite programming approximation to overcome the bilinearities resulting from the multiplication of the asset and the currency returns.
3. Presentation of a hedging strategy that combines robust optimization with the investment in quanto options, which allows for a combined protection of both the asset and the currency risk.
4. Implementation of the suggested models and presentation of numerical results based on real market data. We describe a series of backtesting experiments that compare the proposed models with the Markowitz risk minimization approach.

The rest of this paper is organized as follows. Section 2 formally describes the problem, the convexity issues arising from the multiplication of two random variables, and the solution proposed to overcome these issues by using semidefinite programming. In Section 3 we extend the range of available assets to include quanto options, which provide the investor with an additional insurance regarding any depreciations of foreign exchange rates and asset prices. We implement the proposed models and present numerical results that assess their performance in Section 4. We conclude in Section 5.

2 Robust International Portfolio Optimization

Our starting point is a US investor who wishes to invest in assets from other countries. In order to calculate his returns, he must not only take into account the asset returns in their domestic currency, but also the returns on the foreign exchange rates. We assume that there are n available assets in the market, denominated in m foreign currencies. The current and the future price of the ith asset in its local currency is denoted by P_i^0 and P_i, respectively. The local return of asset i is then $r_i^a = P_i/P_i^0$. We denote by E_j and E_j^0 the future and the current spot exchange rate of the jth currency, respectively. Both quantities are expressed in terms of the base currency per unit of the foreign currency j. The return on a specific currency j is then described by $r_j^e = E_j/E_j^0$. The total return on any asset i will result from the multiplication of the local returns r_i^a with the respective currency returns r_j^e.

Before we are able to formulate the optimization model, we need to define an auxiliary matrix $\boldsymbol{\mathcal{O}}$ that assigns to each asset exactly one currency. If we define o_{ij} as the ijth element of $\boldsymbol{\mathcal{O}}$, then we have:

$$o_{ij} = \begin{cases} 1 & \text{if the } i\text{th asset is traded in the } j\text{th currency} \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

In the Markowitz framework [20] we would want to minimize some risk measure, the portfolio variance, while guaranteeing a minimum expected return, r_{target}. The formulation of our problem would be:

$$\begin{aligned} \min_{w} \quad & \mathbb{E}\left\{[\text{diag}(\boldsymbol{r}^a)\boldsymbol{\mathcal{O}}\boldsymbol{r}^e]'\boldsymbol{w} - \mathbb{E}([\text{diag}(\boldsymbol{r}^a)\boldsymbol{\mathcal{O}}\boldsymbol{r}^e]'\boldsymbol{w})\right\}^{\mathbf{2}} \quad (2) \\ \text{s.t.} \quad & \mathbb{E}([\text{diag}(\boldsymbol{r}^a)\boldsymbol{\mathcal{O}}\boldsymbol{r}^e]'\boldsymbol{w}) \geq r_{\text{target}} \\ & \boldsymbol{w}'\mathbf{1} = 1 \\ & \boldsymbol{w} \geq 0 \end{aligned}$$

where the variable $\boldsymbol{w}$ denotes the vector of asset weights in the portfolio. Throughout this article, variables or parameters in bold face denote vectors. We denote by $\mathbf{1}$ a vector of all ones, whose dimension is clear from the context.

Considering the currency risk in addition to the asset return risk complicates matters, as we are multiplying two random variables. The derivation of the mean and variance of this quadratic function is beyond the scope of this paper. For a complete exposition, we refer the reader to Rustem [22]. Without loss of generality, in our exposition we assume that there are no assets in the domestic currency of the investor. Although this does not alter our discussion, it needs to be taken into account during computation.

2.1 The Robust Model of International Portfolio Optimization

While the Markowitz mean-variance framework has stimulated a significant amount of research and still provides the basis for portfolio management, its assumptions have been subject to criticism. In problem formulation (2), the expected returns have already been estimated and are taken as given. If, however, the materialized returns deviate from the estimates, the determined solution may be far from the optimum or even infeasible. In view of this, we would like to incorporate directly into the model the uncertainty inherent to the estimation of the asset and currency returns. Robust optimization assumes that the returns are random variables, which may materialize in the future within a certain interval. This interval, commonly designated as uncertainty set, reflects the investor's expectations as to how the returns will behave and may be constructed according to some probabilistic measures.

We would like to obtain a solution to our problem that satisfies all the constraints, for all the possible values of the returns within that defined uncertainty set. Hence, we are interested in the worst-case value of the returns for which the solution is still feasible. The robust counterpart of the international portfolio optimization model is:

$$\max_{\boldsymbol{w}} \min_{(\boldsymbol{r}^a,\boldsymbol{r}^e)\in\Xi} [\text{diag}(\boldsymbol{r}^a)\boldsymbol{\mathcal{O}}\boldsymbol{r}^e]'\boldsymbol{w} \quad (3a)$$

$$\text{s.t.} \quad \mathbf{1}'\boldsymbol{w} = 1 \quad (3b)$$

$$\boldsymbol{w} \geq 0 \quad (3c)$$

where we defined the uncertainty set Ξ as:

$$\Xi = \left\{ (\boldsymbol{r^a}, \boldsymbol{r^e}) \geq 0 : \boldsymbol{A r^e} \geq 0 \wedge \left(\begin{bmatrix} \boldsymbol{r^a} \\ \boldsymbol{r^e} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\bar{r}^a} \\ \boldsymbol{\bar{r}^e} \end{bmatrix} \right)' \Sigma^{-1} \left(\begin{bmatrix} \boldsymbol{r^a} \\ \boldsymbol{r^e} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\bar{r}^a} \\ \boldsymbol{\bar{r}^e} \end{bmatrix} \right) \leq \delta^2 \right\} \quad (4)$$

The uncertainty set Ξ defined in (4) results from the intersection of two different sets. The risk associated with the asset and the currency returns is expressed by the uncertainty set:

$$\hat{\Xi} = \left\{ (\boldsymbol{r^a}, \boldsymbol{r^e}) \geq 0 : \left(\begin{bmatrix} \boldsymbol{r^a} \\ \boldsymbol{r^e} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\bar{r}^a} \\ \boldsymbol{\bar{r}^e} \end{bmatrix} \right)' \Sigma^{-1} \left(\begin{bmatrix} \boldsymbol{r^a} \\ \boldsymbol{r^e} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\bar{r}^a} \\ \boldsymbol{\bar{r}^e} \end{bmatrix} \right) \leq \delta^2 \right\}, \quad (5)$$

where we assume that Σ is positive definite. This reflects the idea of a joint confidence interval, where deviations of the returns from their expected values are weighted by the covariance matrix Σ. Note that Σ does not refer only to the relationship between assets, but also between assets and currencies, and between currencies. Indeed, assets and currencies are not thought of as different and separate entities, but their correlation is taken into consideration when optimizing for the optimal portfolio weights:

$$\Sigma = \begin{bmatrix} \Sigma_{r^a} & \Sigma_{r^a r^e} \\ \Sigma'_{r^a r^e} & \Sigma_{r^e} \end{bmatrix} \quad (6)$$

The linear system of inequalities $\boldsymbol{A r^e} \geq 0$ reflects the triangular relationship between the foreign exchange rates, which must be respected at all times to prevent arbitrage. If we define two exchange rates relative to a base currency, for example, the USD versus the EUR (USD/EUR) and the USD versus the GBP (USD/GBP), then we automatically define an exchange rate between the EUR and the GBP as well. When considering that the foreign exchange rate returns may be within a specific interval, we must ensure that the corresponding cross-exchange rate returns are also within adequate intervals. With m foreign currencies in the model, the number of cross exchange rates is $m(m-1)/2$. If we define as X_{jk} the future cross exchange rate between E_j and E_k, that is, X_{jk} is the number of units of currency j that equals one unit of currency k, then:

$$E_j \cdot \frac{1}{E_k} \cdot X_{jk} = 1 \quad (7)$$

In analogy to our previous notation, X^0_{jk} denotes the current spot cross exchange rate, while x_{jk} is the return on the cross exchange rate, that is, X_{jk}/X^0_{jk}. We may modify this equation to express the future exchange rates in terms of the currency returns and the spot exchange rates:

$$\begin{aligned} & E^0_j r^e_j \cdot \frac{1}{E^0_k r^e_k} \cdot X^0_{jk} x_{jk} = 1 \\ \Leftrightarrow \quad & [E^0_j \cdot \frac{1}{E^0_k} \cdot X^0_{jk}] \cdot [r^e_j \cdot \frac{1}{r^e_k} \cdot x_{jk}] = 1 \\ \Leftrightarrow \quad & r^e_j \cdot \frac{1}{r^e_k} \cdot x_{jk} = 1 \end{aligned}$$

Including this constraint, however, will make problem (3) nonconvex. Recall that although we need to model and estimate the future returns of the cross exchange rates, they do not impact our objective function. In fact, their only effect is to constrain further the uncertainty set (4) originally defined for the exchange rates. We express the uncertainty associated with the returns of the cross exchange rates as intervals centered at the estimate, and subsequently make use of the triangular relationship to simplify the expression and eliminate the cross exchange rate returns from the model. Let us assume the cross-exchange rate returns x_{kj} are between a lower and an upper bound, then:

$$\begin{aligned} & L \leq & x_{kj} & \leq U \\ \Leftrightarrow & L \leq & r^e_j / r^e_k & \leq U \\ \Leftrightarrow & L r^e_k \leq & r^e_j & \leq U r^e_k, \end{aligned} \tag{8}$$

which can be expressed as a linear system of inequalities, with matrix $\boldsymbol{A}$ as the respective coefficient matrix.

However, the triangulation requirement is not the only source of nonconvexities in our initial problem formulation (3). Recall that we are multiplying two different sources of returns: the local asset and the currency returns. A common approximation to this problem, initially proposed by Eun and Resnick [9], is to consider the total return on assets as the sum between the local asset returns and the currency returns. In the following subsection, we present an alternative semidefinite programming approach, where a linear function is maximized subject to the constraint that an affine combination of symmetric matrices is positive semidefinite [30].

2.2 Semidefinite Programming Approximation

We start by rewriting our robust problem (3) in the epigraph form [6]:

$$\max_{\boldsymbol{w},\phi} \quad \phi \tag{9a}$$

$$\text{s. t.} \quad [\text{diag}(\boldsymbol{r^a})\boldsymbol{\mathcal{O}r^e}]'\boldsymbol{w} - \phi \geq 0, \quad \forall(\boldsymbol{r^a}, \boldsymbol{r^e}) \in \Xi \tag{9b}$$

$$\boldsymbol{1}'\boldsymbol{w} = 1 \tag{9c}$$

$$\boldsymbol{w} \geq 0, \tag{9d}$$

We show how to replace the semi-infinite inequality constraint (9b) by a linear matrix inequality, using the following result [2]:

Lemma 1. *($\mathcal{S}$-lemma) Given two symmetric matrices $\mathcal{W}$ and $\mathcal{S}$ of the same size and assuming the inequality $\xi'\mathcal{W}\xi \geq 0$ is strictly feasible, that is, $\bar{\xi}'\mathcal{W}\bar{\xi} \geq 0$ for some $\bar{\xi} \in \mathbb{R}^k$, then the following equivalence holds:*

$$[\xi'\mathcal{W}\xi \geq 0 \Rightarrow \xi'\mathcal{S}\xi \geq 0] \Leftrightarrow \exists\lambda \geq 0 : \mathcal{S} \succeq \lambda\mathcal{W}. \tag{10}$$

Lemma 2. *(Approximate $\mathcal{S}$-lemma) Consider t symmetric matrices $\mathcal{W}_l$ with $l = 1, \ldots, t$ and the following propositions:*

(i) $\exists\lambda \in \mathbb{R}^t$ with $\lambda \geq 0$ and $\mathcal{S} - \sum_{l=1}^{t} \lambda_l \mathcal{W}_l \succeq 0$;

(ii) $\xi'\mathcal{S}\xi \geq 0$, $\forall\, \xi \in \Xi := \{\xi \in \mathbb{R}^k : \xi'\mathcal{W}_l\xi \geq 0, l = 1, \ldots, t\}$.

For any $t \in \mathbb{N}$, *(i) implies (ii).*

The proof of Lemma 2 follows along similar lines as **Proposition 3.4** in Kuhn *et al.* [16]. To keep this paper self-contained, we repeat the proof here.

Proof. For any $\xi \in \Xi$, proposition (i) implies that:

$$\xi' \left[\mathcal{S} - \sum_{l=1}^{t} \lambda_l \mathcal{W}_l \right] \xi \quad \geq \quad 0 \tag{11}$$

$$\Leftrightarrow \xi'\mathcal{S}\xi - \sum_{l=1}^{t} \lambda_l \xi'\mathcal{W}_l\xi \quad \geq \quad 0 \tag{12}$$

Because $\lambda \geq 0$ and $\xi'\mathcal{W}_l\xi \geq 0$ for all $\xi \in \Xi$, statement (ii) follows:

$$\xi'\mathcal{S}\xi \geq \xi'\mathcal{S}\xi - \sum_{l=1}^{t} \lambda_l \xi'\mathcal{W}_l\xi \geq 0 \tag{13}$$

□

In order to apply Lemma 2, we rewrite the constraints that define the support of our uncertain returns in the form:

$$\Xi = \{\xi \in \mathbb{R}^k : e_1'\xi = 1,\ \xi'\mathcal{W}_l\xi \geq 0, l = 1, \ldots, t\}, \tag{14}$$

where the first component of the vector ξ is by construction equal to 1. Starting from the uncertainty set $\hat{\Xi}$ in (5), we define an equivalent constraint of the form $\xi'\mathcal{W}_1\xi \geq 0$, where:

$$\xi = \begin{bmatrix} 1 \\ \boldsymbol{r^a} \\ \boldsymbol{r^e} \end{bmatrix}, \quad \mathcal{W}_1 = \begin{bmatrix} (\delta^2 - \begin{bmatrix} \bar{\boldsymbol{r}}^{\boldsymbol{a}\prime} & \bar{\boldsymbol{r}}^{\boldsymbol{e}\prime} \end{bmatrix} \Sigma^{-1} \begin{bmatrix} \bar{\boldsymbol{r}}^{\boldsymbol{a}\prime} & \bar{\boldsymbol{r}}^{\boldsymbol{e}\prime} \end{bmatrix}') & \begin{bmatrix} \bar{\boldsymbol{r}}^{\boldsymbol{a}\prime} & \bar{\boldsymbol{r}}^{\boldsymbol{e}\prime} \end{bmatrix} \Sigma^{-1} \\ \Sigma^{-1} \begin{bmatrix} \bar{\boldsymbol{r}}^{\boldsymbol{a}\prime} & \bar{\boldsymbol{r}}^{\boldsymbol{e}\prime} \end{bmatrix}' & -\Sigma^{-1} \end{bmatrix}$$

A naive incorporation of the triangulation constraint into the new SDP model would imply constructing as many different symmetric matrices as the number of constraints, that is, rows in matrix $\boldsymbol{A}$. We can reduce the number of constraints by expressing the pair of inequalities (8) as a quadratic constraint. We define m_c as the midpoint between Lr_k^e and Ur_k^e, that is, $m_c = (U+L)r_k^e/2$. We note that:

$$Lr_k^e \leq \quad r_j^e \quad \leq Ur_k^e \tag{15a}$$

$$\Leftrightarrow Lr_k^e - m_c \leq \quad r_j^e - m_c \quad \leq Ur_k^e - m_c \tag{15b}$$

$$\Leftrightarrow \quad \frac{L-U}{2} r_k^e \leq \quad r_j^e - m_c \quad \leq \frac{U-L}{2} r_k^e \tag{15c}$$

$$\Leftrightarrow \quad |r_j^e - m_c| \quad \leq \quad \left| \left(\frac{U-L}{2} \right) r_k^e \right|, \tag{15d}$$

where the operator $|\cdot|$ denotes the absolute value. By squaring expression (15d), we may further simplify it to:

$$-UL(r_k^e)^2 - (r_j^e)^2 + (U+L)r_k^e r_j^e \geq 0. \tag{16}$$

For each pairwise inequality in r_j^e and r_k^e, we define the set of constraints $\xi' \mathcal{W}_l \xi \geq 0$, for $l = 2, \ldots, t$, where:

$$\mathcal{W}_l = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Omega_e \end{bmatrix}, \tag{17}$$

with

$$\Omega_e = -(UL)\boldsymbol{e_k}\boldsymbol{e_k}' - \boldsymbol{e_j}\boldsymbol{e_j}' + \frac{1}{2}(U+L)\boldsymbol{e_j}\boldsymbol{e_k}' + \frac{1}{2}(U+L)\boldsymbol{e_k}\boldsymbol{e_j}', \tag{18}$$

where $\boldsymbol{e_k}, \boldsymbol{e_j}$ are the canonical basis vectors in $\mathbb{R}^m$.

We are now able to apply Lemma 2 and replace the inequality constraint (9b) in our original problem with a linear combination of matrices constrained to be positive semidefinite:

$$\max_{\boldsymbol{w},\boldsymbol{\lambda},\phi} \quad \phi \tag{19a}$$

$$\text{s.t.} \quad \mathcal{S} - \sum_{l=1}^{t} \lambda_l \mathcal{W}_l \succeq 0 \tag{19b}$$

$$\boldsymbol{1'w} = 1 \tag{19c}$$

$$\boldsymbol{w}, \boldsymbol{\lambda} \geq 0 \tag{19d}$$

where:

$$\mathcal{S} = \begin{bmatrix} -\phi & 0 & 0 \\ 0 & 0 & \frac{1}{2}\text{diag}(\boldsymbol{w})\boldsymbol{\mathcal{O}} \\ 0 & \frac{1}{2}\boldsymbol{\mathcal{O}}'\text{diag}(\boldsymbol{w}) & 0 \end{bmatrix}$$

The reformulated problem (19) on the decision variables $\boldsymbol{w}$ and $\boldsymbol{\lambda}$ constitutes a conservative approximation, that is, it provides a lower bound to our original problem (9). A similar procedure to compute lower bounds has been suggested by Shor and others, see [30]. Moreover, the semidefinite program is a convex optimization problem, as both its objective function and constraints are convex. We have therefore eliminated the intractability issues in our model and we are able to solve it efficiently with a modern semidefinite programming solver such as SDPT3 [26, 29].

Our investor may also wish to add a further constraint to guarantee a minimum expected return. Recall that we are multiplying two random variables, and without any further assumptions, the expected value of the product of two random variables is not necessarily the product of each variable expected value. Rustem [22] provides a computationally tractable approach to evaluate the mean and variance of the product of two random variables. We follow his approach to compute the expected value of the portfolio:

$$\mathbb{E}([\text{diag}(\boldsymbol{r^a})\boldsymbol{\mathcal{O}}\boldsymbol{r^e}]'\boldsymbol{w}) \geq r_{\text{target}} \tag{20a}$$

$$\Leftrightarrow [\text{diag}(\boldsymbol{\bar{r}^a})\boldsymbol{\mathcal{O}}\boldsymbol{\bar{r}^e}]'\boldsymbol{w} + \frac{1}{2}\text{trace}(\Sigma\Omega) \geq r_{\text{target}} \tag{20b}$$

where

$$\Omega = \begin{bmatrix} 0 & \text{diag}(\boldsymbol{w})\mathcal{O} \\ \mathcal{O}'\text{diag}(\boldsymbol{w}) & 0 \end{bmatrix}$$

Maximizing the portfolio return in view of the worst possible outcomes of the asset and the currency returns ensures that the investor receives a guaranteed wealth at maturity date. In fact, as long as any variation of the asset and the currency returns stays within the boundaries of the uncertainty set Ξ, the investor will obtain a portfolio return higher than (or in the worst case equal to) the value of the objective function determined in (19) — that is the *non-inferiority* property of robust optimization. In the next section, we look at introducing quanto options as another investment alternative. These options are able to provide the investor not only with an additional profit, but also with a protection against both a depreciation of the foreign exchange rates and a decrease of the local asset returns. One of the disadvantages of the proposed robust optimization approach is that it provides a guarantee only *inside* the uncertainty set, while no insurance is provided if the returns materialize *outside* the uncertainty set. Because put options are able to lock a certain value of the foreign exchange rates and of the asset prices, they can provide a protection outside the uncertainty set considered.

3 Downside Risk Protection with Quanto Options

Robust optimization provides a guaranteed minimum portfolio return as long as the returns remain within the uncertainty set considered. Unless the worst case materializes, the investor will always obtain a better portfolio return. Options may provide additional guarantees as they allow the investor to lock in a specific foreign exchange rate or asset price.

In the international portfolio optimization literature, the hedging instrument typically used is the forward contract, despite this being a binding agreement where a specific amount of money will be exchanged, thus not offering the investor any flexibility. Recent works have studied the performance of options as a hedging instrument. However, results seem to point towards the better performance of forwards when compared to simple options strategies, see [27].

3.1 Hedging with Quanto Options

Options are a flexible instrument as they give their buyer the right but not the obligation to buy (*call*) or sell (*put*) another asset, called the underlying, at a future date for a specified price, the strike [15]. While Giddy [11] argues that options are a better suited instrument for situations in which the amount to be received in the future is uncertain, Steil [25] states that in the case of foreign investments, currency options are not suitable as the underlying asset does not correspond to the contingency that we wish to hedge against. In an international portfolio, if the investor wishes to be protected against both depreciations of the foreign exchange rate and losses in the value of the assets, he would have to buy both currency and equity options. We propose to use quanto options to overcome these issues. Quanto options or "quantity-adjusting options" are mostly used in foreign exchange markets, where the price of an underlying asset needs to be converted into another underlying asset at a fixed guaranteed rate, [31]. In a study by Ho *et al.* [14], it is shown that quanto put options provide a better downside protection as they take into account the correlation between the asset and the foreign exchange rate.

In our modelling framework, we follow closely the approach suggested by Zymler [32]. We define the payoff of a quanto put option Q as the difference between the strike price K and the spot price of the underlying asset P at maturity date, translated to the base currency of the investor at a specified exchange rate $\bar{E}$:

$$Q = \max\left\{0, \bar{E}(K - P)\right\} \tag{21}$$

Note that both the strike price K and the price of the asset P are denominated in foreign currency and translated at the fixed foreign exchange rate $\bar{E}$ expressed in units of the base currency per unit of the foreign currency. The exchange rate chosen is usually the forward rate with the same maturity as the option. In 1992, Reiner [21] formally derived a pricing formula for quanto options in the domestic currency, based on the same assumptions as the Black & Scholes model [5]. The key aspect of his formulation lies in the inclusion of the correlation coefficient ρ between the foreign equity and the exchange rate. We define the premium p^q of a quanto put option with expiration date in T periods of time as:

$$p^q = \bar{E}\left\{Ke^{-rT}N(\sigma_s\sqrt{T} - d_1) - Pe^{(r_f - r - \rho\sigma_s\sigma_{fx})T}N(d_1)\right\} \tag{22}$$

where:

$$d_1 = \frac{\log{(P/K)} + (r_f - \rho\sigma_s\sigma_{fx} + \sigma_s^2/2)T}{\sigma_s\sqrt{T}}, \tag{23}$$

σ_s and σ_{fx} denote the standard deviation of the asset price and the foreign exchange rate respectively, $N(\cdot)$ is the standard normal distribution, and r and r_f are the domestic and the foreign risk-free rate respectively. We concentrate solely on the payoff and pricing functions of put options, as our model will only include put options. The inclusion of call options could easily be done following the same approach as for put options. Because we are interested in the potential hedging benefits of options, we choose to include only put options.

In order to include quanto options in our robust optimization model, we define as r^q_{ij} the return on the jth quanto option on the ith foreign asset, given that there are k options available for each asset:

$$r^q_{ij} = \max\left\{0, \frac{\bar{E}\left(K_{ij} - P_i\right)}{p^q_{ij}}\right\} \tag{24}$$

Without loss of generality, we assume that each asset has the same number of options available in the market. The future spot price P_i of the underlying asset may be rewritten as a function of the return on the ith asset r^a_i and the asset's spot price P^0_i:

$$r^q_{ij} = \max\left\{0, \frac{\bar{E}\left(K_{ij} - P^0_i r^a_i\right)}{p^q_{ij}}\right\} \tag{25}$$

As in the previous section, we wish to maximize our portfolio return in view of the worst-case of the asset and the currency returns, assuming that these will materialize in the uncertainty set defined in (4). A new vector of weights $\boldsymbol{w_q}$ defines the percentage of the budget allocated to quanto put options. We formulate our hedging model as:

$$\max_{\boldsymbol{w},\boldsymbol{w_q},\phi} \quad \phi \tag{26a}$$

$$\begin{aligned}
\text{s. t. } [\text{diag}(\boldsymbol{r^a})\boldsymbol{\mathcal{O}}\boldsymbol{r^e}]'\boldsymbol{w} + \boldsymbol{r^q}'\boldsymbol{w_q} - \phi &\geq 0, \forall(\boldsymbol{r^a}, \boldsymbol{r^e}) \in \Xi, \boldsymbol{r^q} = f(\boldsymbol{r^a}) & (26b)\\
\boldsymbol{1}'\boldsymbol{w} + \boldsymbol{1}'\boldsymbol{w_q} &= 1 & (26c)\\
\boldsymbol{w}, \boldsymbol{w_q} &\geq 0 & (26d)
\end{aligned}$$

Note that $\boldsymbol{r^q}$, though dependent on two parameters, is interpreted as a vector. Writing the return on the quanto options $\boldsymbol{r^q}$ as a function $f(\cdot)$ of the local asset returns $\boldsymbol{r^a}$, implies that constraint (26b) must be satisfied for all the random returns in Ξ, plus:

$$r_{ij}^q \geq \frac{\bar{E}\left(K_{ij} - P_i^0 r_i^a\right)}{p_{ij}^q}, \ \forall\, i = 1, \ldots, n, \ \forall\, j = 1, \ldots, k \tag{27}$$

$$r_{ij}^q \geq 0, \qquad \forall\, i = 1, \ldots, n, \ \forall\, j = 1, \ldots, k \tag{28}$$

Again, we will use Lemma 2 to derive an equivalent tractable formulation to the hedging problem (26). As in Section 2, we rewrite the constraints referring to the quanto options in the quadratic form:

$$\xi_q' \mathcal{W}_l \xi_q \geq 0, \text{ with } l = 1, \ldots, 2(kn),$$

where the vector ξ_q is augmented by the variables $\boldsymbol{r^q}$:

$$\xi_q' = \begin{bmatrix} 1 & \boldsymbol{r^a} & \boldsymbol{r^e} & \boldsymbol{r^q} \end{bmatrix}.$$

This is possible because the first element of the Ξ vector is 1.

Note that there are k options for each asset of the n assets, and that for each option a symmetric matrix $\mathcal{W}$ on the returns and on the non-negativity constraint must be considered. Therefore the total number of new matrices to be introduced in the model amounts to $2(kn)$. Given that the vector ξ_q has been augmented by (kn) new variables, the symmetric matrices regarding the uncertainty set (4) must also reflect this change and be augmented by (kn) rows and columns.

With these additional matrices, we are now able to replace the semi-infinite inequality (26b) in our hedging model (26) with a linear combination of matrices constrained on their positive semidefiniteness:

$$\max_{\boldsymbol{w},\boldsymbol{w_q},\boldsymbol{\lambda},\phi} \quad \phi \tag{29a}$$

$$\begin{aligned}
\text{s. t. } \quad \mathcal{S} - \sum_{l=1}^{t} \lambda_l \mathcal{W}_l &\succeq 0 & (29b)\\
\boldsymbol{1}'\boldsymbol{w} + \boldsymbol{1}'\boldsymbol{w_q} &= 1 & (29c)\\
\boldsymbol{w}, \boldsymbol{w_q}, \boldsymbol{\lambda} &\geq 0 & (29d)
\end{aligned}$$

where:

$$\mathcal{S} = \begin{bmatrix} -\phi & 0 & 0 & \frac{1}{2}\boldsymbol{w_q'} \\ 0 & 0 & \frac{1}{2}\text{diag}(\boldsymbol{w})\mathcal{O} & 0 \\ 0 & \frac{1}{2}\mathcal{O}'\text{diag}(\boldsymbol{w}) & 0 & 0 \\ \frac{1}{2}\boldsymbol{w_q} & 0 & 0 & 0 \end{bmatrix}$$

Note that the consideration of a large number of options may result in numerical problems during computation, as for each option considered, we include two new matrices $\mathcal{W}_l$. Moreover, increasing the number of matrices can have an adverse effect on the quality of the approximation to the optimal solution.

As before, we could also include a constraint on the expected portfolio return:

$$\mathbb{E}([\text{diag}(\boldsymbol{r^a})\boldsymbol{\mathcal{O}r^e}]'\boldsymbol{w}) \geq r_{\text{target}} \tag{30}$$

We do not include the option returns in this constraint, as we are only interested in their hedging potential and not on speculating with options. Given their leverage effect and the fact that we optimize for the worst-case of the currency and the asset returns, if no restrictions were set on the options' weight, the optimal solution would be to invest the full budget on "in-the-money" options. With our approach however, the investor guarantees a certain expected portfolio return resulting from the asset returns and is able to invest the remainder budget in options, if that is the optimal solution.

In model (29) we use options as an additional mean to optimize for the worst-case return within the uncertainty set. We now elaborate a model where options are used in order to limit the investor's exposure to market realisations outside the uncertainty set:

$$\max_{\boldsymbol{w},\boldsymbol{w_q},\phi} \quad \phi \tag{31a}$$

$$\begin{aligned}
\text{s.t.}\quad [\text{diag}(\boldsymbol{r^a})\boldsymbol{\mathcal{O}r^e}]'\boldsymbol{w} - \phi &\geq 0, \quad \forall(\boldsymbol{r^a},\boldsymbol{r^e}) \in \Xi & \text{(31b)}\\
[\text{diag}(\boldsymbol{r^a})\boldsymbol{\mathcal{O}r^e}]'\boldsymbol{w} + \boldsymbol{r^q}'\boldsymbol{w_q} - \beta\phi &\geq 0, \ \forall \boldsymbol{r^a},\boldsymbol{r^e} \geq 0,\ \boldsymbol{r^q} = f(\boldsymbol{r^a}) & \text{(31c)}\\
\boldsymbol{1}'\boldsymbol{w} + \boldsymbol{1}'\boldsymbol{w_q} &= 1 & \text{(31d)}\\
\boldsymbol{w},\boldsymbol{w_q} &\geq 0, & \text{(31e)}
\end{aligned}$$

Constraint (31c) imposes the portfolio value comprised of both assets and options to be greater than a percentage β of the worst-case return, when the random returns materialize outside the uncertainty set.

By investing in put options, the investor is able to lock in a certain price of the underlying asset denominated in his domestic currency. In the case of foreign assets, the exchange risk is effectively eliminated by considering a fixed foreign exchange rate in which to translate the respective payoff. Depending on the total investment in options, in particular relative to the amount invested in foreign assets, the investor may benefit from an increased protection even when the asset returns fall outside the uncertainty set considered. In the next section, we perform a series of experiments following the implementation of both the robust and the hedging models.

4 Numerical Results

The theoretical framework developed in Sections 2 and 3 will now be used to compute optimal solutions to our international portfolio model. We assume the point of view of a US investor who wishes to invest not only in domestic assets, such as the S&P500 and the NASDAQ, but also in foreign assets. We consider 3 international indexes, namely, the German DAX and the French

CAC40, both denominated in EUR, and the Swiss SMI in CHF. The models were implemented in YALMIP [19] together with the semidefinite programming solver SDPT3 [26, 29]. The expected returns on the local assets and on the foreign exchange rates, as well as the covariance matrix are constructed from 10 years of monthly data between October 1998 and September 2008, see Table 1.

	Ret. (%)	Std. (%)	Correl.						
DAX	3.37	23.40	1.00						
CAC40	3.40	18.64	0.92	1.00					
SMI	2.39	14.76	0.77	0.82	1.00				
S&P500	2.09	14.14	0.79	0.77	0.72	1.00			
NASDAQ	2.93	27.79	0.71	0.67	0.48	0.83	1.00		
EUR	2.15	8.28	-0.14	-0.19	-0.19	0.01	0.02	1.00	
CHF	2.34	8.55	-0.23	-0.27	-0.28	-0.06	-0.05	0.94	1.00

Table 1: Distributional parameters of annual returns (Oct-98 to Sep-08)

4.1 Portfolio Composition

Throughout this section, we will designate problem (19) as the robust problem, and problem (29) as the hedging problem. We first would like to measure the impact of the size of the uncertainty set on the chosen assets and then assess how the introduction of options influences these choices.

The size of the uncertainty set, defined by δ^2, is a subjective parameter, which depends on the risk aversion of the investor. Mainly, the uncertainty set should reflect the investor's expectations of the future returns, and it can be constructed according to some probabilistic measures of the returns distribution. If we assume the future returns are normally distributed and δ^2 is assigned the value of the αth percentile of a χ^2 distribution with v degrees of freedom, then there is a probability of $\alpha\%$ that the future returns will be inside the uncertainty set [7]:

$$\text{if } \delta^2 = (\chi^2)^{-1}(\alpha\%, v) \Rightarrow \text{Prob}((\boldsymbol{r^a}, \boldsymbol{r^e}) \in \Xi) \geq \alpha\%$$

The degrees of freedom should be equal to the number of random returns in the portfolio.

We first compare the portfolio composition between the Markowitz model and the robust model. We find that both models invest in the same assets, but with different weights. We also measure the impact of increasing the size of the uncertainty set on the worst-case return in the case of the robust model, see Figure 1. As expected, the higher the value of δ, the smaller the worst-case return, that is, our robust model is only able to guarantee the investor with a smaller return. We also note that when an expected return constraint is included, the worst-case return is smaller. This difference is more accentuated for higher values of δ. As we now impose a return constraint, weights are allocated differently in order to satisfy this constraint, therefore the optimizer is not able to guarantee the same return anymore.

We now include a further guarantee for the investor in the form of options, and assess how that impacts the portfolio composition. We have included five different options for each asset: simple put options for the domestic assets and

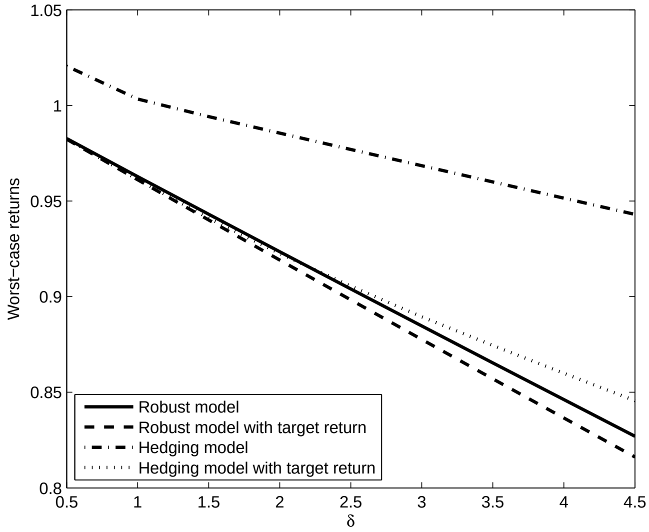

Figure 1: Relationship between worst-case return and parameter δ

quanto put options for the foreign assets. The considered strike prices are at the maximum distance of 10% from the current spot price of the underlying asset, and are equally distant from each other. We have therefore included two options "in-the-money", two options "out-of-the-money", and one option "at-the-money". The option prices are calculated according to the pricing formula proposed by Reiner [21] in the case of quanto options, see Section 3. Standard put option prices were calculated according to the Black & Scholes model [5].

Figure 1 depicts the impact of a larger uncertainty set, that is, higher values of δ, on the worst-case portfolio return, when options are included as an alternative investment strategy. The first thing to note is that the worst-case return with options in the portfolio is always higher than or equal to the one without options, irrespective of the size of the uncertainty set. Investing in options thus provide an additional guarantee in the form of a lower bound on the portfolio value. Because we do not impose any restriction on the hedging strategy, options may be bought even when there are no portfolio holdings on the respective underlying asset. In the case of our particular data set and if we do not consider a constraint on the expected return, the weight allocated to put options decreases for larger uncertainty set sizes, from 28%, when $\delta = 0.5$, to 15% when $\delta = 4.5$, concentrating on "in-the-money" put options. Also, if we add an expected return constraint, investment in options decreases considerably, weighing only about 1% of the portfolio. Moreover, the options chosen to invest in are either "at-the-money" or "out-of-the-money", given that additional resources have to be allocated to the assets in order to satisfy the expected return constraint.

The choice for "in-" and "at-the-money" put options, though it may seem surprising, can be explained by the fact that we are not optimizing for the worst-case of the options returns but only for the worst-case of the assets and the currencies returns. If that was the case, the optimal put option would have a strike price equal or very close to the worst-case asset price. With our formulation, the chosen options provide not only a hedging guarantee, but also a profit opportunity.

4.2 Model Evaluation with Historical Market Prices

We want to evaluate the performance of both the robust and the hedging models under real market conditions and over a long period of time. To this end, we consider the real index returns and the respective real currency returns in the period from October 1998 until September 2008. Each month we calculate the optimal asset allocation taking the expected asset and currency returns as the mean of the historical returns from the previous twelve months. The upper and lower bounds of the cross-exchange rates were calculated based on their mean returns for the period considered plus the standard deviation for the same period multiplied by a factor of ± 1.5. These bounds and the covariance matrix Σ are assumed to remain constant throughout this period. At the end of each month, the actual portfolio return is computed based on the materialized returns, and the options (if any) are exercised or left to expiry depending on the spot price of the asset. This procedure is repeated every month, and the accumulated wealth is calculated.

We consider five different options for each asset in the portfolio. In the case of domestic assets, simple put options are included, while for foreign assets, we include quanto options. Because quanto options are mainly traded over-the-counter, there are no records of historical premiums. In order to perform our backtesting experiment, we simulate the options premiums based on the pricing formula developed by Reiner [21] described in Section 3. For the simple put options, we use the Black & Scholes model [5]. We consider five different strike prices in the range of 10% equally distant from the current asset price. The fixed foreign exchange rate $\bar{E}$ is assumed to be the historical forward exchange rate with one month maturity, equal to the option maturity. We consider an annual risk-free rate of 3.32% for the US investor (based on LIBOR annual rates for the same period).

We have solved the robust and the hedging models over the considered period for different sizes of the uncertainty set δ, computed the cumulative gains and compared them with the results obtained from the Markowitz risk minimization model. We have also imposed an expected return constraint of 5% per year. Recall that this expected return must originate only from the asset returns, which prevents the entire budget from being allocated to options.

Figure 2 depicts the accumulated wealth from October 1998 to September 2008 for the three different models. For this particular data set and parameter choice, the minimum risk model is outperformed by the robust model, while the hedging model dominates both the robust and the minimum risk models. The average annual returns for the robust and the hedging models are 6% and 9% respectively, while the Markowitz model only provides a return of 2.84%. We have also computed the average annual return for different values of the parameter δ for both the robust and the hedging models, see Table 2.

Because we are optimizing for the worst-case scenario, our accumulated portfolio returns, even in times of decreasing asset prices, are never as low as in the Markowitz model. From January 2002 until September 2003, both the

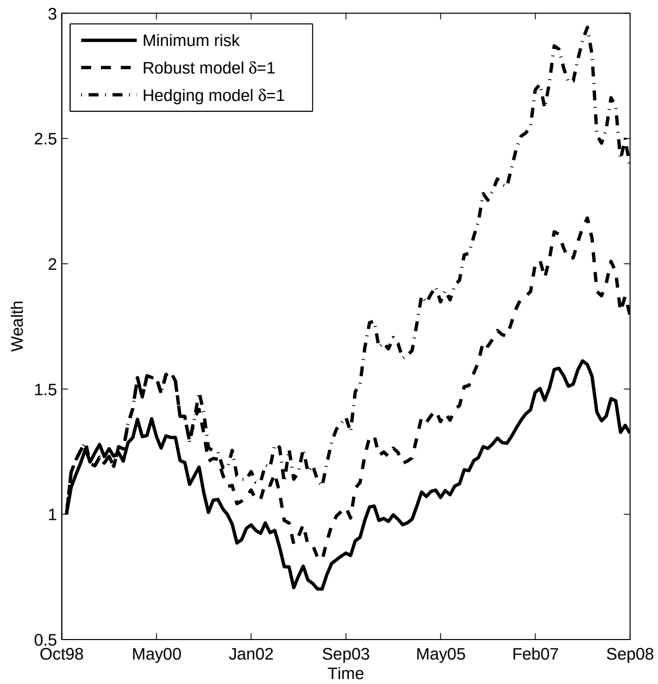

Figure 2: Accumulated wealth over the period from Oct98 to Sep08

δ	Robust Ret. (%)	Hedging Ret. (%)
0.5	6.47	8.31
1	6.02	9.03
1.5	5.98	8.89
2	6.06	9.51
2.5	6.01	11.66
3	5.99	12.56
3.5	5.98	12.80

Table 2: Average annual returns for different values of parameter δ

Markowitz risk minimization model and the robust model incurred in losses, though not so significant in the latter case. The insurance effect of the put options is clearly seen in that same period. The hedging model is the only model that guarantees an accumulated wealth above 1, that is, without any losses to the investor.

We are also interested in assessing the performance of the hedging model in the case where options provide an additional guarantee against the future returns materializing outside the uncertainty set. We have conducted the backtesting experiment described above for the alternative hedging model (31). We compared the backtesting results of the robust model with those of the new hedging formulation for different values of β, namely $\beta \in \{0.3; 0.5; 0.7; 0.9\}$, where $\delta = 1$. We found that there was no improvement from the robust model when including this new constraint, as all of the curves coincided, yielding an average annual return of about 6%. We note that with this new formulation the optimal solution is to invest in "out-of-the-money" options with the strike price as close as possible to the worst-case asset price. This results differs from our previous results where preference was given to "at-the-money" options. The frequency of "out-of-the-money" options being exercised is below 50%, therefore in many cases they represent only a cost. Although potentially more expensive as it relies on "at-the-money" options, our initial hedging formulation (29) allows not only for a hedging guarantee, but also for a profit opportunity. Exercised put options also offer a protection in case the asset and the currency returns fall outside the uncertainty set, and in this way the additional constraint is redundant.

Although the backtesting results seem to point towards a good performance of the hedging model, these results should also be regarded with caution. Because we use simulated option prices, there is a risk of underestimating these prices, which favors the investment in options and could cause an upward-bias of the results. Furthermore, we have not considered the risk of default from the writer of the option, which in the case of over-the-counter traded options might be significant.

5 Conclusion

In this paper, we extend the paradigm of robust optimization to the international portfolio allocation problem. We show that, although the naive problem formulation is nonconvex due to the multiplication of asset and currency returns, it has a tractable convex formulation. The model we obtain by employing the approximate S-Lemma is a conservative approximation to our original problem. We further extend the robust optimization approach by complementing it with an investment in quanto options as an additional insurance. Quanto options link a foreign equity option with a forward rate, and they have been shown to be more effective in downside risk protection than the separate consideration of foreign equity and currency options.

The suggested approach can be considered to be more flexible than the standard hedging strategies, as it relies on options and robust optimization, and not exclusively on forward rates. Furthermore, the hedging strategy is implemented from a portfolio perspective and does not depend on the future value of any particular asset or currency. The backtesting results seem to point towards the better performance of the robust model when compared to the classical Markowitz risk minimization model. The hedging model with options outperforms both the robust and the risk minimization models in the considered data set.

References

[1] A. Ben-Tal, L. E. Ghaoui, and A. Nemirovski. *Robust Optimization*. Princeton University Press, 2009.

[2] A. Ben-Tal, A. Goryashko, E. Guslitzer, and A. Nemirovski. Adjustable robust solutions of uncertain linear programs. *Mathematical Programming Ser. A*, 99:351–376, 2004.

[3] A. Ben-Tal and A. Nemirovski. Robust convex optimization. *Mathematics of Operations Research*, 23:769–805, 1998.

[4] F. Black. Universal hedging: Optimizing currency risk and reward in international equity portfolios. *Financial Analysts Journal*, 45:16–22, 1989.

[5] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–654, 1973.

[6] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge University Press, 2004.

[7] S. Ceria and R. A. Stubbs. Incorporating estimation errors into portfolio selection: Robust portfolio construction. *Journal of Asset Management*, 7:109–127, 2006.

[8] L. El-Ghaoui and H. Lebret. Robust solutions to least-squares problems with uncertain data. *SIAM Journal on Matrix Analysis & Applications*, 18:1035–1064, 1997.

[9] C. S. Eun and B. G. Resnick. Exchange rate uncertainty, forward contracts, and international portfolio selection. *The Journal of Finance*, XLIII:197–215, 1988.

[10] R. Fonseca, S. Zymler, W. Wiesemann, and B. Rustem. Robust optimization of currency portfolios. Available on COMISEF Working Paper Series (WPS-012).

[11] I. H. Giddy. Foreign exchange options. *The Journal of Futures Markets*, 3:143–166, 1983.

[12] J. Glen and P. Jorion. Currency hedging for international portfolios. *The Journal of Finance*, XLVIII:1865–1886, 1993.

[13] H. Grubel. Internationally diversified portfolios: Welfare gains and capital flows. *American Economic Review*, 58:1299–1314, 1968.

[14] T. S. Ho, R. C. Stapleton, and M. G. Subrahmanyam. Correlation risk, cross-market derivative products and portfolio performance. *European Financial Management*, 1:105–124, 1995.

[15] J. C. Hull. *Options, Futures and Other Derivatives*. Pearson International Edition, 2006.

[16] D. Kuhn, W. Wiesemann, and A. Georghiou. Primal and dual linear decision rules in stochastic and robust optimization. *Mathematical Programming (Online First)*.

[17] G. A. Larsen and B. G. Resnick. The optimal construction of internationally diversified equity portfolios hedged against exchange rate uncertainty. *European Financial Management*, 6:479–514, 2000.

[18] H. Levy and M. Sarnat. International diversification of investment portfolios. *The American Economic Review*, 60:668–675, 1970.

[19] J. Löfberg. Yalmip: A toolbox for modeling and optimization in MATLAB. In *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004.

[20] H. Markowitz. Portfolio selection. *Journal of Finance*, 7:77–91, 1952.

[21] E. Reiner. *From Black-Scholes to Black Holes - New Frontiers in Options*, chapter Quanto Mechanics, pages 147–156. RISK, 1992.

[22] B. Rustem. Computing optimal multi-currency mean-variance portfolios. *Journal of Economics Dynamics & Control*, 19:901–908, 1995.

[23] B. Rustem and M. Howe. *Algorithms for Worst-Case Design and Applications to Risk Management.* Princeton University Press, 2002.

[24] H. A. Shawky, R. Kuenzel, and A. D. Mikhail. International portfolio diversification: a synthesis and an update. *Journal of International Financial Markets, Institutions & Money*, 7:303–327, 1997.

[25] B. Steil. Currency options and the optimal hedging of contingent foreign exchange exposure. *Economica*, 60:413–431, 1993.

[26] K. Toh, M. Todd, and R. Tutuncu. SDPT3 - a MATLAB software package for semidefinite programming. *Optimization Methods and Software*, 11:545–581, 1999.

[27] N. Topaloglou, H. Vladimirou, and S. A. Zenios. Controlling currency risk with options or forwards. *Working Papers, HERMES European Center of Excellence on Computational Finance and Economics*, pages 1–27, 2007.

[28] N. Topaloglou, H. Vladimirou, and S. A. Zenios. A dynamic stochastic programming model for international portfolio management. *European Journal of Operational Research*, 185:1501–1524, 2008.

[29] R. Tutuncu, K. Toh, and M. Todd. Solving semidefinite-quadratic-linear programs using SDPT3. *Mathematical Programming Ser. B*, 95:189–217, 2003.

[30] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38:49–95, 1996.

[31] P. G. Zhang. *Exotic Options: a guide to second generation options.* World Scientific, 1998.

[32] S. Zymler, B. Rustem, and D. Kuhn. Robust portfolio optimization with derivative insurance guarantees. Available on www.optimization-online.org.