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Pair-Copula Selection with Downside Risk Minimization

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Pair-Copula Selection with Downside Risk Minimization

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Abstract

Copulae provide investors with tools to model the dependency structure among financial products. The choice of copulae plays an important role in successful copula applications. However, selecting copulae usually relies on general goodness-of-fit (GoF) tests which are independent of the particular financial problem. This paper first proposes a pair-copula-GARCH model to construct the dependency structure and simulate the joint returns of five U.S. equities. It then discusses copula selection problem from the perspective of downside risk management with the so-called D-vine structure, which considers the Joe-Clayton copula and the Student t copula as building blocks for the vine pair-copula decomposition. Value at risk, expected shortfall, and Omega function are considered as downside risk measures in this study. As an alternative to the traditional bootstrap approaches, the proposed pair-copula-GARCH model provides simulated asset returns for generating future scenarios of portfolio value. It is found that, although the Student t pair-copula system performs better than the Joe-Clayton system in a GoF test, the latter is able to provide the loss distributions which are more consistent with the empirically examined loss distributions while optimizing the Omega ratio. Furthermore, the economic benefit of using the pair-copula-GARCH model is revealed by comparing the loss distributions from the proposed model and the conventional exponentially weighted moving average model of RiskMetrics in this case.

Key words. Downside Risk, AR-TGARCH, Pair-Copula, Vine Structure, Differential Evolution.

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1 Introduction

A knowledge of dependence structure of financial products has become increasingly important in major financial applications, such as portfolio management, risk management and financial derivative pricing. Traditional mean-variance portfolio theory does not consider the nonlinear and asymmetrical dependence of asset returns as the theory assumes multivariate returns being normally distributed (see Markowitz [1952]). When underlying returns follow a multivariate normal distribution, the Pearson correlation coefficient is sufficient to describe the dependence between risk factors. However, the multivariate normal distribution assumption has been challenged in practice. It is widely acknowledged in the literature that this assumption does not follow the empirical evidence, e.g. the stylised facts introduced by Cont [2001]. Patton [2004] found that both the skewness of individual asset returns and the asymmetry in the dependence between stocks were economically significant and statistically significant. The work of Rachev et al. [2005] also provides empirical evidence rejecting the hypothesis that returns for financial products are normally distributed and instead shows that the returns exhibit fat tails and skewness. These stylised facts are important to portfolio management. For instance, the diversification effect may be overstated if portfolio managers ignore the nonlinear and asymmetrical dependence structure.

Sklar [1959] proposed a solution for modeling dependence structure of random variables, i.e. the copula, which isolates dependence structure from univariate marginal distributions to formulate multivariate distributions. Nelsen [1998] further provided a comprehensive introduction of the copula theory. In the literature, copulae have been widely applied by market practitioners to model dependence structure of financial risk factors, and most of the copula applications have focused on modeling bivariate distributions. For example, Embrechts et al. [1999] proposed copulae as descriptions of dependence between financial risk factors, and Cherubini et al. [2004] discussed various applications of the copula theory to financial problems. Patton [2006] further proposed extensions of the copula theory to allow for conditioning variables, and employed it to construct flexible models of the conditional dependence structure of exchange rates. Ammann and Süss [2009] applied the Skewed t copula to generate meta Skewed Student t distributions, and it was found that the asymmetry property of the copula helped to improve the description of dependence structure between equities returns.

Copula functions also have been drawn attention to modeling high-dimensional distributions. A recent study by Fischer et al. [2009] shows that vine pair-copula decompositions may be more appropriate for modeling high-dimensional distributions than other approaches including the multivariate Archimedean copulae, the Koehler-Symanowski copulae and the Multiplicative Liebscher copulae. However, it is important that the choice of pair-copula should be considered when using the vine pair-copula decompositions. There are two popular types of parametric copulae: the elliptical copulae, which are extracted from elliptical distributions, and the Archimedean copulae, which are constructed by using generators from the Archimedean copula families. Fischer et al. [2009] suggested that the Student t copula should be the preferred one in most financial applications. However, the study of Fischer et al. [2009] included the one-parameter Archimedean copulae, it might have been better if the study had considered the mixture of one-parameter Archimedean copulae or the two-parameter Archimedean copulae.

As investors may prefer portfolios which are designed for risk minimization, several downside risk measures (e.g. Value-at-Risk (VaR), Expected Shortfall (ES) and Omega ratio) have been considered as an alternative to the variance measure by financial practitioners in the past decade. Distributing weights to minimize loss probability of having returns under a given level of risk measures has been applied to both passive and active portfolio management. For passive portfolio management, similar asset allocation problems can be dated back to some studies over 50 years ago. For example, Roy [1952] discussed the optimum distributions with the so-called 'safety first' rule when portfolio returns were assumed to be normally distributed. Rockafellar and Uryasev [2000] studied resource allocation problems with VaR or ES minimizations provided that the loss functions were convex and continuously differentiable. Gilli et al. [2006] discussed portfolio selection problems which were designed for minimizing downside risks subject to certain real-world constraints. Vassiliadis et al. [2009] propose a hybrid ant colony optimization algorithm for active portfolio management under a downside risk framework. Copula theory has been restored to estimate these risk measures in portfolio management. Most of the VaR-copula studies reveal that the portfolios which consider the nonlinear and asymmetric dependence structure are more robust than those constructed under the assumption of multivariate normal distribution (see Embrechts et al. [1999] and Bradley and Taqqu [2004]).

This paper makes two contributions to the literature. First, it combines pair-copula decompositions with GARCH models to construct the dependence structure and simulate the joint returns of five equities. The marginal distributions are modeled by using AR-TGARCH models which allow asymmetric effects from past innovations to affect the conditional variance (see Rabemananjara and Zakoian [1993]), and the innovations are modeled by using the Skewed Student t distribution of Hansen [1994]. The multivariate dependence structure of the innovations is constructed by using a D-vine pair-copula decomposition proposed by Bedford and Cooke [2002]. The economic benefit of using the proposed model is illustrated by the accuracy of simulated loss distribution based on simulated returns from the model, in the comparison with the losses based on return simulation from other conventional econometric models, such as the exponentially weighted moving average (EWMA) model of RiskMetrics [1996].

Secondly, this paper discusses the copula selection problem for the vine pair-copula decomposition from the perspective of downside risk management. Aas et al. [2007] suggested plotting the original data of each bivariate case or performing goodness-of-fit (GoF) tests to decide the choice of pair-copula. However, relying on visual plots may only lead to a rough guess as to the dependence structure, and existing statistical GoF tests can only distinguish suitable copulae for modeling the multivariate distributions, rather than choose the most appropriate copula for a specified financial application. Especially, selecting an optimal copula arises when examined copula models are able to pass GoF tests with high p-values. Since financial practitioners usually need to choose only one copula model for a specified problem, this paper provides a supplementary discussion to the copula selection problem under such circumstances.

The structure of this paper is as follows. Section 2 introduces pair-copula decompositions for multivariate dependence construction and Section 3 presents AR-TGARCH models for modeling marginal distributions. Section 4 gives an asset allocation problem with downside risk minimization. Section 5 provides the results of the experiment and a discussion. Section 6 summarizes the paper.

2 Pair-Copula Decompositions with Vine Structures

2.1 Pair-Copula Decompositions

Bedford and Cooke [2002] suggested decomposing a multivariate density into a product of marginal densities and conditional densities. The latter can be written recursively by using a so-called pair-copula decomposition. Considering a d-dimensional vector $\boldsymbol{X} = (X_1, \ldots, X_d)$, the joint density distribution can be expressed as

$$f(x_1, \ldots, x_d) = f(x_d) \cdot f(x_{d-1}|x_d) \cdot f(x_{d-2}|x_{d-1}, x_d) \cdot \ldots f(x_1|x_2, \ldots, x_d). \quad (1)$$

As Sklar [1959] introduced, any multivariate distribution H with marginal densities $F_1(x_1), \ldots, F_d(x_d)$ may be written as

$$H(x_1, \ldots, x_d) = C\{F_1(x_1), \ldots, F_d(x_d)\}, \quad (2)$$

where C denotes a d-dimensional copula, representing a multivariate distribution with uniformly distributed marginals U on $[0, 1]$. Therefore, the copula in Eq. (2) could be written as

$$C(u_1, \ldots, u_d) = H\{F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\}, \quad (3)$$

in which $F_j^{-1}(u_j)$ is the inverse cumulative distribution function of the j-th marginal density.

The joint density of the copula function can be found by differentiating Eq. (3)

$$f(x_1, \ldots, x_d) = c_{1,\ldots,d}\{F_1(x_1), \ldots, F_d(x_d)\} \cdot f_1(x_1) \cdots f_d(x_d), \tag{4}$$

where $c_{1,\ldots,d}(\cdot)$ denotes a d-dimensional copula density. Then one can obtain the conditional densities. In a bivariate case, the conditional density can be written as

$$f(x_1|x_2) = c_{1,2}\{F_1(x_1), F_2(x_2)\} \cdot f_1(x_1), \tag{5}$$

where $c_{1,2}$ is the so-called pair-copula density for the two transformed variables. The conditional density in a multivariate case can be found as

$$f(x|\mathbf{v}) = c_{x,v_j|\mathbf{v}_{-j}}\{F(x|\mathbf{v}_{-j}), F(v_j|\mathbf{v}_{-j})\} \cdot f(x|\mathbf{v}_{-j}), \tag{6}$$

where v_j denotes an arbitrarily chosen component of $\mathbf{v}$, and $\mathbf{v}_{-j}$ represents the $\mathbf{v}$ vector excluding the j-th component. Joe [1996] showed that $F(x|\mathbf{v})$ could be computed by using

$$F(x|\mathbf{v}) = \frac{\partial C_{x,v_j|\mathbf{v}_{-j}}\{F(x|\mathbf{v}_{-j}), F(v_j|\mathbf{v}_{-j})\}}{\partial F(v_j|\mathbf{v}_{-j})}. \tag{7}$$

However, the pair-copula decompositions in high-dimensional cases (e.g. $d \geq 3$) are not unique. For example, there are two possible pair-copula densities for a 3-dimensional case:

$$f(x_1|x_2, x_3) = c_{1,2|3}\{F(x_1|x_3), F(x_2|x_3)\} \cdot f(x_1|x_3), \quad \text{or} \tag{8}$$
$$f(x_1|x_2, x_3) = c_{1,3|2}\{F(x_1|x_2), F(x_3|x_2)\} \cdot f(x_1|x_2). \tag{9}$$

In order to arrive at a unique decomposition, the so-called vines, e.g. the D-vine and the canonical vine were introduced by Bedford and Cooke [2002] and Kurowicka and Cooke [2005] to graphically describe the decomposition scheme of vine structures. The vines actually generalize the Markov trees which have been used within the area of uncertainty analysis to build up high-dimensional dependent distributions (see Cooke et al. [1991]. As the D-vine structure has been successfully applied and recommended by researchers for modeling equity returns in high-dimensional cases (see Aas et al. [2007] and Fischer et al. [2009]), this structure is adopted in this paper. The D-vine structure is shown in Figure 1. The figure describes a 5-dimensional structure which comprises four chains $\Upsilon_j, j = 1, \ldots, 4$. Each chain has $6 - j$ nodes and $5 - j$ edges. The edge represents

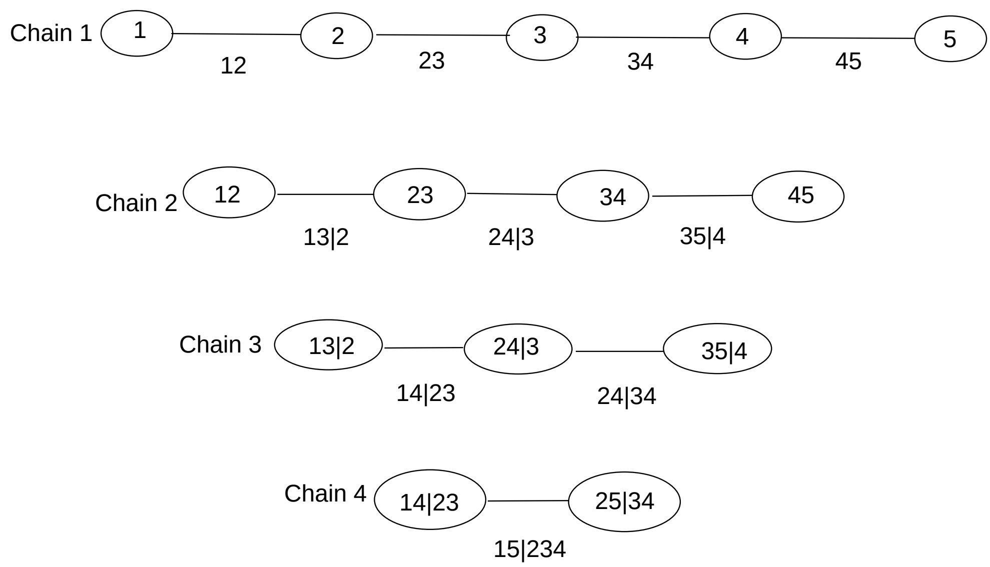


Figure 1: The Five-Dimensional D-vine Structure
(reproduced from Aas et al. [2007])

a pair-copula density, and the edge label corresponds to the subscript in the pair-copula density. In total, a number of $d(d-1)/2$ bivariate copulae with a number of d marginal densities jointly define the multivariate density.

The density of a d-dimensional distribution with the D-vine pair-copula decomposition can be found in Aas et al. [2007] as

$$\begin{aligned} f(x_1,\dots,x_d) = \prod_{k=1}^{d} f(x_k) \prod_{j=1}^{d-1}\prod_{i=1}^{d-j} & c_{i,i+j|i+1,\dots,i+j-1} \\ & \{F(x_i|x_{i+1},\dots,x_{i+j-1}), F(x_{i+j}|x_{i+1},\dots,x_{i+j-1})\}. \end{aligned} \tag{10}$$

In the 5-dimensional case, the copula density with the D-vine decomposition is written as

$$\begin{aligned} & f(x_1,x_2,x_3,x_4,x_5) = \\ & \qquad f_5(x_5)\cdot f(x_4|x_5)\cdot f(x_3|x_4,x_5)\cdot f(x_2|x_3,x_4,x_5)\cdot f(x_1|x_2,x_3,x_4,x_5). \end{aligned} \tag{11}$$

The conditional densities in Eq. (11) can be further decomposed by using the pair-copula densities with the marginal distributions

$$f(x_4|x_5) = c_{45}\{F_4(x_4),F_5(x_5)\}\cdot f_4(x_4) \tag{12}$$

$$\begin{aligned} f(x_3|x_4,x_5) = & c_{35|4}\{F(x_3|x_4),F(x_5|x_4)\} \\ & \cdot c_{34}\{F_3(x_3),F_4(x_4)\}\cdot f_3(x_3) \end{aligned} \tag{13}$$

$$\begin{aligned} f(x_2|x_3,x_4,x_5) = & c_{25|34}\{F(x_2|x_3,x_4),F(x_5|x_3,x_4)\} \\ & \cdot c_{24|3}\{F(x_2|x_3),F(x_4|x_3)\}\cdot c_{23}\{F_2(x_2),F_3(x_3)\}\cdot f_2(x_2) \end{aligned} \tag{14}$$

$$\begin{aligned} f(x_1|x_2,x_3,x_4,x_5) = & c_{15|234}\{F(x_1|x_2,x_3,x_4),F(x_5|x_2,x_3,x_4)\} \\ & \cdot c_{14|23}\{F(x_1|x_2,x_3),F(x_4|x_2,x_3)\} \\ & \cdot c_{13|2}\{F(x_1|x_2),F(x_3|x_2)\}\cdot c_{12}\{F_1(x_1),F_2(x_2)\}\cdot f_1(x_1). \end{aligned} \tag{15}$$

The conditional cumulative distributions in Eq. (13) to Eq. (15) can be further decomposed by using Eq. (7)

$$F(x_3|x_4) = \frac{\partial C_{34}\{F_3(x_3),F_4(x_4)\}}{\partial F_4(x_4)} \tag{16}$$

$$F(x_5|x_4) = \frac{\partial C_{45}\{F_4(x_4),F_4(x_5)\}}{\partial F_4(x_4)}, \tag{17}$$

$$F(x_2|x_3,x_4) = \frac{\partial C_{24|3}\{F(x_2|x_3),F(x_4|x_3)\}}{\partial F(x_4|x_3)} \tag{18}$$

$$F(x_2|x_3) = \frac{\partial C_{23}\{F_2(x_2),F_3(x_3)\}}{\partial F_3(x_3)} \tag{19}$$

$$F(x_4|x_3) = \frac{\partial C_{34}\{F_3(x_3),F_4(x_4)\}}{\partial F_3(x_3)} \tag{20}$$

$$F(x_5|x_3,x_4) = \frac{\partial C_{35|4}\{F(x_4|x_5),F(x_3|x_4)\}}{\partial F(x_3|x_4)} \tag{21}$$

$$F(x_5|x_4) = \frac{\partial C_{45}\{F_4(x_4),F_5(x_5)\}}{\partial F_4(x_4)} \tag{22}$$

$$F(x_3|x_4) = \frac{\partial C_{34}\{F_3(x_3),F_4(x_4)\}}{\partial F_4(x_4)}, \tag{23}$$

$$F(x_1|x_2,x_3,x_4) = \frac{\partial C_{14|13}\{F(x_1|x_2,x_3),F(x_4|x_2,x_3)\}}{\partial F(x_4|x_2,x_3)} \tag{24}$$

$$F(x_1|x_2,x_3) = \frac{\partial C_{13|2}\{F(x_1|x_2),F(x_3|x_2)\}}{\partial F(x_3|x_2)} \tag{25}$$

$$F(x_1|x_2) = \frac{\partial C_{12}\{F_1(x_1),F_2(x_2)\}}{\partial F_2(x_2)} \tag{26}$$

$$F(x_3|x_2) = \frac{\partial C_{23}\{F_2(x_2),F_3(x_3)\}}{\partial F_2(x_2)} \tag{27}$$

$$F(x_4|x_2,x_3) = \frac{\partial C_{24|3}\{F(x_2|x_3),F(x_4|x_3)\}}{\partial F(x_2|x_3)} \tag{28}$$

$$F(x_2|x_3) = \frac{\partial C_{23}\{F_2(x_2),F_3(x_3)\}}{\partial F_3(x_3)} \tag{29}$$

$$F(x_4|x_3) = \frac{\partial C_{34}\{F_3(x_3),F_4(x_4)\}}{\partial F_3(x_3)}, \tag{30}$$

$$F(x_5|x_2,x_3,x_4) = \frac{\partial C_{25|34}\{F(x_2|x_3,x_4),F(x_5|x_3,x_4)\}}{\partial F(x_2|x_3,x_4)} \tag{31}$$

$$F(x_2|x_3,x_4) = \frac{\partial C_{24|3}\{F(x_2|x_3),F(x_4|x_3)\}}{\partial F(x_4|x_3)} \tag{32}$$

$$F(x_2|x_3) = \frac{\partial C_{23}\{F_2(x_2),F_3(x_3)\}}{\partial F_3(x_3)} \tag{33}$$

$$F(x_4|x_3) = \frac{\partial C_{34}\{F_3(x_3),F_4(x_4)\}}{\partial F_3(x_3)} \tag{34}$$

$$F(x_5|x_3,x_4) = \frac{\partial C_{35|4}\{F(x_3|x_4),F(x_5|x_4)\}}{\partial F(x_3|x_4)} \tag{35}$$

$$F(x_3|x_4) = \frac{\partial C_{34}\{F_3(x_3),F_4(x_4)\}}{\partial F_4(x_4)} \tag{36}$$

$$F(x_5|x_4) = \frac{\partial C_{45}\{F_4(x_4),F_5(x_5)\}}{\partial F_4(x_4)}. \tag{37}$$

To check whether the pair-copula decomposition is suitable for modeling the dependence structure of a specified data set, Aas et al. [2007] performed a GoF test for the vine pair-copula based on a probability integral transform (PIT) which was suggested by Rosenblatt [1952]. The PIT transforms a set of d-dimensional dependent variables X_i into a new set of variables $\mathfrak{X}_i$ which are supposed to be independent and uniformly distributed on $[0,1]^d$. To verify whether the transformed variables are independent and uniformly distributed, a new variable $\mathfrak{R} = \sum_{i=1}^{d}\{\Phi^{-1}(\mathfrak{X}_i)\}^2$ is introduced, and the null hypothesis that $\mathfrak{R}$ follows a χ^2 distribution with d DoF is tested. Only this GoF test is considered in this paper since the main interest of the paper focuses on financial applications rather than statistical studies. For a more sophisticated GoF tests reference should be made to the study of Genest et al. [2009].

Inference of the pair-copula parameters and simulation of the random numbers with the D-vine structure need the conditional cumulative copula functions and its inverse functions. The following subsections introduce the two functions of the Student t copula and the Joe-Clayton copula. This paper follows the parameter inference and random number simulation processes suggested by Aas et al. [2007].

2.2 The Student t Copula

The bivariate Student t copula has the form

$$C^t_{\nu,\boldsymbol{\rho}}(\mathbf{u}) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{\Gamma(\frac{\nu+2}{2})}{\Gamma(\frac{\nu}{2})\sqrt{(\pi\nu)^2|\boldsymbol{\rho}|}} \left(1+\frac{\mathbf{x}^\top\boldsymbol{\rho}\mathbf{x}}{\nu}\right)^{-\frac{\nu+2}{2}} dx, \tag{38}$$

where u_i is the value after taking x_i as the input of its cumulative probability function. After taking the derivatives $c^t(u_1, u_2) = \frac{\partial^2 C^t(u_1,u_2)}{\partial u_1 \partial u_2}$, one has the bivariate Student t copula density:

$$c^t_{\nu,\boldsymbol{\rho}}(u_1, u_2) = \frac{1}{\sqrt{|\boldsymbol{\rho}|}} \frac{\Gamma(\frac{\nu+2}{2})\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu+1}{2})^2} \frac{\prod_{k=1}^{2}(1+\frac{t_\nu^{-1}(u_k)^2}{\nu})^{\frac{\nu+1}{2}}}{(1+\frac{(t_\nu^{-1}(\mathbf{u}))'\boldsymbol{\rho}^{-1}(t_\nu^{-1}(\mathbf{u}))}{\nu})^{\frac{\nu+2}{2}}}, \tag{39}$$

where ν denotes the DoF of the Student t copula. $\Gamma(\cdot)$ is the Gamma function, and $\boldsymbol{\rho}$ represents the correlation coefficient matrix.

The conditional cumulative function on u_2 of the bivariate Student t copula is defined as

$$h^t(u_1, u_2; \nu, \rho) = t_{\nu+1}\left(\frac{t_\nu^{-1}(u_1) - \rho \cdot t_\nu^{-1}(u_2)}{\sqrt{\frac{(\nu+(t_\nu^{-1}(u_2))^2)(1-\rho^2)}{\nu+1}}}\right). \tag{40}$$

The inverse of the h^t function is given by Aas et al. [2007]

$$h^{-t}(u_1, u_2; \nu, \rho) = t_\nu\left(t_{\nu+1}^{-1}(u_1) \cdot \sqrt{\frac{(\nu+(t_\nu^{-1}(u_2))^2)(1-\rho^2)}{\nu+1}} + \rho \cdot t_\nu^{-1}(u_2)\right). \tag{41}$$

2.3 The Joe-Clayton Copula

The Joe-Clayton copula belongs to the two-parameter families of Archimedean copulae (BB7 in Joe [1997]), and it has the form

$$\begin{aligned} C^{jc}(u_1, u_2; \tau^U, \tau^L) &= 1 - \left(1 - \left((1-(1-u_1)^\kappa)^{-\gamma} + (1-(1-u_2)^\kappa)^{-\gamma} - 1\right)^{-1/\gamma}\right)^{1/\kappa} \\ \kappa &= 1/(\log_2(2-\tau^U)) \\ \gamma &= -1/(\log_2 \tau^L). \end{aligned} \tag{42}$$

τ^L and τ^U denote the lower and the upper dependence measures respectively. After differentiating the copula function, one has the Joe-Clayton copula density function

$$c^{jc}(u_1, u_2; \tau^U, \tau^L) = \frac{R}{Q^2} \cdot \frac{(1-Q^{-1/\gamma})^{1/\kappa}}{(Q^{1/\gamma}-1)^2} \cdot \left\{-1 + \kappa \cdot \left(Q^{1/\gamma} + \gamma \cdot (Q^{1/\gamma} - 1)\right)\right\} \tag{43}$$

$$R = (1-(1-u_1)^\kappa)^{-\gamma-1}(1-u_1)^{\kappa-1}(1-(1-u_2)^\kappa)^{-\gamma-1}(1-u_2)^{\kappa-1} \tag{44}$$

$$Q = \left\{-1 + (1-(1-u_1)^\kappa)^{-\gamma} + (1-(1-u_2)^\kappa)^{-\gamma}\right\}. \tag{45}$$

The cumulative function of the copula conditioning on u_2 is written as

$$
\begin{aligned}
h^{jc}(u_1, u_2; \tau^U, \tau^L) = & \Big\{1 - \big\{-1 + (1-(1-u_1)^\kappa)^{-\gamma} + (1-(1-u_2)^\kappa)^{-\gamma}\big\}^{-1/\gamma}\Big\}^{1/\kappa - 1} \\
& \cdot \big\{-1 + (1-(1-u_1)^\kappa)^{-\gamma} + (1-(1-u_2)^\kappa)^{-\gamma}\big\}^{-1/\gamma - 1} \\
& \cdot \{1-(1-u_2)^\kappa\}^{-\gamma-1} \cdot (1-u_2)^{\kappa-1}. \qquad (46)
\end{aligned}
$$

Bisection method, which is a numerical method to find roots of equations, is employed to solve the inverse of the $h^{jc}(\cdot)$ function with respect to the first variable u_1, i.e. the inverse of the conditional distribution function. For a general introduction of the bisection method and its applications can be found in Burden and Faires [2005] and Berry and Zuo [2009].

3 AR-Threshold-GARCH Model for Margins

Over the last few decades, GARCH models proposed by Bollerslev [1987] and Engle [1982] have been applied by researchers to characterize the stylised facts in asset returns. For example, the multivariate GARCH models have been suggested in modeling high-dimensional distributions for risk management, optimal hedging and contagion study (see Bollerslev et al. [1988], Hansson and Hordahl [1998], and Bae et al. [2003]). However, Ang and Bekaert [2002] and Ang and Chen [2002] reported that the multivariate GARCH models might not well describe the tail dependence structure. As an alternative, copula-GARCH methods are recommended by researchers to model multivariate distributions. When applying copula-GARCH models, the univariate residuals from GARCH models are usually explained by using approaches such as empirical distribution functions, kernel functions, fat tail distributions or the Extreme Value Theory.

One of the advantages of using GARCH models is that the extensive framework of GARCH-type models, such as the GJR-GARCH, the FI-GARCH and the threshold-GARCH (TGARCH), is able to capture the well-known return trait of different financial products. In the literature, the AR-TGARCH models have been successfully employed by researchers to model return of financial products (see Jondeau and Rockinger [2006], Floros [2007] and Lai et al. [2009]). Based on preliminary experiments, the return mean functions of the equities are modeled by using AR processes in this paper. Let the returns of an asset be given by r_t, the AR-TGARCH model is defined by

$$
\begin{aligned}
r_t =& \varphi_0 + \varphi_1 r_{t-1} + \varphi_2 r_{t-2} + \cdots + \varepsilon_t, && (47)\\
\varepsilon_t =& \sigma_t \varsigma_t, && (48)\\
\sigma_t^2 =& \alpha_0 + \alpha_1^+(\varepsilon_{t-1}^+)^2 + \alpha_1^-(\varepsilon_{t-1}^-)^2 + \beta_1 \sigma_{t-1}^2, && (49)\\
\varsigma_t \sim& \mathrm{SkT}(\eta, \lambda). && (50)
\end{aligned}
$$

Eq. (47) defines the conditional mean function with parameters φ_t of the AR processes and the innovation ε_t. As the lag number of the AR processes depends on individual equities, it is not specified in Eq. (47). Eq. (48) defines the innovation ε_t as the product of the conditional volatility, σ_t and the residual, ς_t. The dynamics of volatility is given by Eq. (49) with the notation of $\varepsilon_t^+ = \max(\varepsilon_t, 0)$ and $\varepsilon_t^- = \max(-\varepsilon_t, 0)$. Due to the positivity and the stationarity constraints of the volatility process, the TGARCH parameters should satisfy the constraints: $\alpha_0, \alpha_1^+, \alpha_1^-, \beta_1 \geq 0$, and $(\alpha_1^+ + \alpha_1^-)/2 + \beta_1 < 1$ as Glosten et al. [1993] suggested. Eq. (50) specifies the residuals by using a Skewed Student t distribution, which was defined by Hansen [1994]

$$\mathrm{SkT}(\dot{y};\eta,\lambda) = \begin{cases} bc(1+\frac{1}{\eta-2}(\frac{b\dot{y}+a}{1-\lambda})^2)^{-(\eta+1)/2} & \text{if } \dot{y} < -a/b, \\ bc(1+\frac{1}{\eta-2}(\frac{b\dot{y}+a}{1+\lambda})^2)^{-(\eta+1)/2} & \text{if } \dot{y} \geq -a/b, \end{cases} \tag{51}$$

where

$$a \equiv 4\lambda c\frac{\eta-2}{\eta-1}, \quad b^2 \equiv 1+3\lambda^2-a^2, \quad c \equiv \frac{\Gamma(\frac{\eta+1}{2})}{\sqrt{\pi(\eta-2)}\Gamma(\frac{\eta}{2})}. \tag{52}$$

λ represents the asymmetry parameter, and $\dot{y}$ follows the standard Student t distribution with a marginal DoF η.

The $\boldsymbol{\varsigma}$ of each return series are taken as the marginal observations, i.e. the $\boldsymbol{u}$ in the Student t copula model and the Joe-Clayton copula model introduced in Section 2.2 and Section 2.3 respectively. In this paper, the two-step maximum likelihood method which has been studied and discussed by Genest et al. [1995], is employed to estimate the parameters of copula models.

4 Portfolio Construction with Downside Risk Minimization

4.1 Optimization Problem

The optimization problem for asset selection with downside risk minimization has been discussed by Gilli et al. [2006], whereas this paper addresses the copula selection with the asset allocation problem. At time t_0, there is an initial wealth B_0 to be invested in a set of N assets with prices $S_{i,0}$, $i = 1, ..., N$. At the beginning of the planning investment horizon t_0, the asset returns r_i and the portfolio value P_T at the end of the horizon are unknown, whereas they may be estimated by employing simulation studies.

To relax the normality assumption of the joint returns under the Markowitz framework, the future portfolio value P_T is estimated based on simulated returns, i.e. a set of possible realizations of returns, which are generated by using the proposed pair-copula-GARCH model. In other words, the possible realization of returns $r_{\mathfrak{s},i}$, with $\mathfrak{s} = 1, ..., n_{\mathfrak{S}}$ and $i = 1, ..., N$ are simulated by using the pair-copula-GARCH model, where $n_{\mathfrak{S}}$ denoting the number of simulated scenarios. Thus, the portfolio value of each simulation can be written as

$$P_{\mathfrak{s},T} = \sum_{i}^{N} n_i \cdot S_{i,0} \cdot (1 + r_{\mathfrak{s},i}), \qquad \mathfrak{s} = 1, ..., n_{\mathfrak{S}}, \tag{53}$$

where n_i represents the share number invested in the i-th stock.

The loss of the portfolio from each simulation run is defined as

$$\mathfrak{L}_{\mathfrak{s}} = P_0 - P_{\mathfrak{s},T}, \qquad \mathfrak{s} = 1, ..., n_{\mathfrak{S}}. \tag{54}$$

As Gilli et al. [2006] suggested, the $\text{VaR}_{1-\alpha}$ estimated from the simulated scenarios can be written as the $(1-\alpha)$-th loss of the $n_{\mathfrak{S}}$ simulated losses in ascending order such that $\mathfrak{L}_1 \leq \mathfrak{L}_2, ... \leq \mathfrak{L}_{n_{\mathfrak{S}}}$, then the VaR can be found as

$$\text{VaR} = \mathfrak{L}_{(\lceil (1-\alpha) \cdot n_{\mathfrak{S}} \rceil)}, \tag{55}$$

where $1 - \alpha$ is the probability that the loss will not exceed the VaR value.

The ES can be computed as

$$\text{ES} = \frac{1}{\sum \mathbf{1}_{\{\mathfrak{L}_{\mathfrak{s}} > \text{VaR}\}}} \sum \mathfrak{L}_{\mathfrak{s}} \mathbf{1}_{\{\mathfrak{L}_{\mathfrak{s}} > \text{VaR}\}}. \tag{56}$$

The Omega function, which was proposed by Keating and Shadwick [2002], can be estimated as

$$\text{Omega} = -\frac{\sum \mathfrak{L}_{\mathfrak{s}} \mathbf{1}_{\{\mathfrak{L}_{\mathfrak{s}} > 0\}}}{\sum \mathfrak{L}_{\mathfrak{s}} \mathbf{1}_{\{\mathfrak{L}_{\mathfrak{s}} < 0\}}}. \tag{57}$$

The asset allocation problem can be formulated as

$$\min_{\mathbf{n}} \mathfrak{O}(\mathfrak{L}) \tag{58}$$

$$n_i \in \mathbb{N}_0^+ \tag{59}$$

$$P_0 = \sum_{i}^{N} n_i \cdot S_{i,0} = \text{B}_0 \tag{60}$$

$$w^{min} \leq \frac{n_i \cdot S_{i,0}}{P_0} \leq w^{max}, \tag{61}$$

where $\mathfrak{O}(\cdot)$ is the objective function representing the risk measure which is defined in Eq. (55) to Eq. (57), and $\mathbf{n}$ represents the vector of share numbers of each equity invested. Eq. (59) denotes an integer constraint on the share numbers. Eq. (60) is the budget constraint, or the sum-to-one constraint. In other words, the shares of invested equities comprise a portfolio with a market value of $P_0 = \sum_{i=1}^{N} n_i \cdot S_{i,0} = \mathrm{B}_0$, i.e., $\sum_{i=1}^{N} w_i = 1$ with $w_i = \frac{n_i \cdot S_{i,0}}{\mathrm{P}_0}$. Eq. (61) imposes a weight constraint on the holding size of an asset with $w^{min} = 1\%$ and $w^{max} = 50\%$. The initial budget B_0 was set at \$1,000,000.

Given that the pair-copula-GARCH model provides reliable joint asset return simulations, (i) the portfolio weights should agree with the weights optimized based on empirically observed return simulations (i.e. the bootstrapped returns), and (ii) the loss distribution should be consistent with the empirically examined one, no matter which risk measure is used.

4.2 Optimization Method

Evolutionary methods, such as Genetic Algorithm, Threshold Accepting and Differential Evolution, have been used to tackle the complex optimization problems in finance and economics (see Winker [2001] and Gilli et al. [2008]). Constraints, such as the lots constraint, can be tackled by using these evolutionary methods. This paper employs Differential Evolution (DE) (see Storn and Price [1997]) to solve the optimization problem. DE initializes its population by using random numbers. For each current solution $\boldsymbol{\imath}_{\mathfrak{p}}$, a new solution $\boldsymbol{\imath}_{\mathfrak{c}}$ is generated from the following process. First, the algorithm randomly selects three different chromosomes from the current population ($\mathfrak{p}_1 \neq \mathfrak{p}_2 \neq \mathfrak{p}_3 \neq \mathfrak{p}$). Then genes of the new chromosome are generated by linearly combining the genes from the chromosomes at a probability π_1, otherwise inheriting the genes of the original $\mathfrak{p}$-th solution. Extra noises are considered to escape from local optima and avoid premature convergence. In this paper, vectors $\boldsymbol{z_1}$ and $\boldsymbol{z_2}$ represent the extra noises. The two vectors contain random numbers being zero at the probabilities $\boldsymbol{\pi_2}$ and $\boldsymbol{\pi_3}$ respectively, or being normally independent distribution $N(0, \mathfrak{d}_1^2)$ and $N(0, \mathfrak{d}_2^2)$ otherwise. The linear combination can be described as:

$$\imath_{\mathfrak{c}}[i] := \begin{cases} \imath_{\mathfrak{p}_1}[i] + (K + z_1[i]) \cdot (\imath_{\mathfrak{p}_2}[i] - \imath_{\mathfrak{p}_3}[i] + z_2[i]) & \text{with probability } \pi_1 \\ \imath_{\mathfrak{p}}[i] & \text{otherwise,} \end{cases}$$

where π_1 is the crossover probability. After the linear combination, DE updates the population. More specifically, if the fitness value of $\boldsymbol{\imath}_{\mathfrak{c}}$ is higher than the one of $\boldsymbol{\imath}_{\mathfrak{p}}$, then $\boldsymbol{\imath}_{\mathfrak{p}}$ is replaced by $\boldsymbol{\imath}_{\mathfrak{c}}$, and the updated $\boldsymbol{\imath}_{\mathfrak{p}}$ exists in the current population, otherwise the original $\boldsymbol{\imath}_{\mathfrak{p}}$ survives. Since the solutions from DE may be either positive or negative, the no-short-selling constraint might be violated if one directly interprets $\boldsymbol{\imath}_{\mathfrak{p}}$ as portfolio weights. A mapping function is used to translate the solution into asset weights. The assets are first assigned with the minimum weight w^{min}, and then the weights are increased in proportion to the values in $\boldsymbol{\imath}_{\mathbf{p}}$ until the sum of weights add up to unity. If an equity weight exceeds the maximum weight, its weight is decreased to w^{max}, and the excess part is superadded proportionally to other equities according to their weights. The holding of the i-th equity $n_i = \lceil w_i P_0 / S_{i,0} \rceil$ is computed by rounding up to the closest integer. The mapping function has been employed by Maringer and Oyewumi [2007] for index tracking.

The technical parameters of DE algorithm are listed as follows. Population size and iteration number were set at 50 and 500 respectively. The value of K was set at a value 0.5 and the crossover probability π_1 was set at 60%. The parameters used for generating the noise vectors were $\pi_2 = 50\%$, $\pi_3 = 10\%$, $\mathfrak{d}_1^2 = 0.1$ and $\mathfrak{d}_2^2 = 0.1$.

5 The Experiments

5.1 Data

The portfolio comprises five equities selected from the top ten S&P 500 stocks: Johnson & Johnson (J&J), Cisco Systems, Bank of America (BoA), General Electric (GE), and AT&T. These stocks are considered as representatives of five sectors: healthcare, information technology, finance, industrials and telecommunications. Daily log-return of the five assets in the period 3 January 2000 to 1 July 2009 are plotted in Figure 2.

Table 1 summarizes some preliminary descriptive statistics of the daily returns. Most of the average returns are negative, except for J&J. The unconditional standard deviations reveal that J&J is the least volatile of the five equities during the period. The skewness (SK) and the excess kurtosis (eKU) indicate that the five return distributions are asymmetric and fat-tailed. The autocorrelation and the heteroscedasticity of the five return series are revealed by using the Ljung-Box and the ARCH Lagrange multiplier (LM) tests respectively. The statistics from the Ljung-Box test indicate that the squared return series are autocorrelated up to lag 10, and the ARCH LM test statistics show the presence of the autoregressive conditional heteroscedasticity in the five return series up to order 10. The lower panel of the table reports the unconditional correlation coefficients of the five stock returns.

A bootstrap approach, the exponentially weighted moving average (EWMA) model, and the proposed pair-copula GARCH model were used separately to simulate the daily returns of the five equities. The bootstrap approach was adopted to generate return simulations with a bootstrap number of $n_{\mathfrak{S}} = 20,500$. In each simulation, a block size of 20 returns was randomly drawn from the set of 824 daily returns in the period 3 January 2006 to 30 June 2009. The planned investment

Table 1: Summary Statistics on the Five Daily Returns

	J&J	Cisco	BoA	GE	AT&T
Mean	0.0001	-0.0004	-0.0002	-0.0005	-0.0003
Max	0.1154	0.2182	0.3021	0.1798	0.1508
Min	-0.1725	-0.1405	-0.3421	-0.1368	-0.1354
SD	0.0146	0.0311	0.0361	0.0229	0.0208
SK	-0.6018	0.3322	-0.2921	0.0915	0.1345
eKU	14.5283	5.1659	22.8774	7.2292	4.3939
Q(1)	22.5018	54.0363	237.3108	148.6092	31.0346
p-values	0.0000	0.0000	0.0000	0.0000	0.0000
Q(10)	262.1906	497.9050	1,377.3619	1,093.9157	559.5125
p-values	0.0000	0.0000	0.0000	0.0000	0.0000
LM(1)	22.4624	53.9455	236.8924	148.7618	30.9800
p-values	0.0000	0.0000	0.0000	0.0000	0.0000
LM(10)	145.3386	217.6292	472.3026	438.0548	257.3098
p-values	0.0000	0.0000	0.0000	0.0000	0.0000
J&J	1	0.185033	0.267299	0.373999	0.346643
Cisco	0.185033	1	0.332776	0.458931	0.326552
BoA	0.267299	0.332776	1	0.575315	0.360903
GE	0.373999	0.458931	0.575315	1	0.404473
AT&T	0.346643	0.326552	0.360903	0.404473	1

period considered in this paper is one month. The sum of the daily log-return of each block defines a monthly log-return. This return simulation follows the approach of Gilli et al. [2006].

A rolling window strategy was adopted to simulate joint asset returns by using the proposed pair-copula GARCH model. As using GARCH models usually requires large data samples, the parameters of the pair-copula GARCH model were estimated based on the daily returns of a six-year period, i.e. the rolling window starts from January 2000 with a window length of $\mathfrak{w}_{\mathfrak{l}} = 1,500$. The daily return simulations were generated by using the pair-copula GARCH model with the estimated parameters at a monthly interval. For example, the daily observations from 2 January 2000 to 30 December 2005 were first used to infer the parameters of the pair-copula-GARCH model. Then the possible realizations of daily asset returns in January 2006 were simulated by using the pair-copula-GARCH model with the estimated parameters. The six-year window rolls at an interval of 20 observations, i.e. $\wp = 20$, roughly representing a monthly frequency. Thus, there were 41 updates over the period 3 January 2006 to 30 June 2009. A block size of

20 returns with a number of 500 were simulated by using the pair-copula-GARCH model for each month. Consequently, the simulated returns consist of a total simulation scenarios of $n_{\mathfrak{S}} = 20,500$ with a block size of 20 in the period 3 January 2006 to 30 June 2009. Figure 3 briefly describes the simulation process.

In addition to the bootstrap approach and the pair-copula-GARCH model, a block size of 20 daily returns for the five assets were simulated by using the EWMA model with an iteration number of 500 at the beginning of each month from 3 January 2006 to 30 June 2009. The simulated daily returns were generated based on the recent 250 historical observations with a decay factor at 0.94 (see RiskMetrics [1996]). Consequently, the number of simulated scenarios of the block returns was 20,500 over the examined period in this case.

5.2 Estimation of the Marginal Models

Figure 4 provides the estimated parameters of the marginal AR-TGARCH models for the five equities based on the six-year monthly rolling window. It is found that most of the parameters are volatile during the U.S. subprime and the recent financial crisis. Table 2 reports the parameters and statistics of the marginal AR-TGARCH models from the last update (i.e. the period July 2004 to June 2009). The first part of Table 2 shows a clear asymmetric volatility responding to the positive and negative innovations in the five return series. Although the parameters satisfy the constraint $(\hat{\alpha}_1^+ + \hat{\alpha}_1^-)/2 + \hat{\beta}_1 < 1$, the volatility of most stocks (e.g. J&J, BoA and GE) tends to follow an explosive GARCH process when the past innovation is negative, which implies that negative innovations may lead to temporal instability.

The marginal DoF parameter $\hat{\eta}$ and the asymmetry parameter $\hat{\lambda}$ of the Skewed Student t distributions are reported below the TGARCH parameters. The DoF parameters of the five equities have a range of 4 to 8, implying that the normal distribution assumption is inappropriate in modeling the residuals. Although the asymmetry parameters $\hat{\lambda}$ are not different from 0 at the 5% significance level, it has been decided to include the asymmetry parameter in the model after applying the two-sample Kolmogorov-Smirnov (KS) test. The empirically observed residuals were compared with a hypothesized distribution which was constructed by using the Student t CDF with the marginal parameters (indicated by KS test^a). Then the same empirically observed residuals were compared with another hypothesized distribution which was constructed by using the Skewed Student t CDF with the estimated parameters (indicated by KS test^b). Interestingly, all of the p-values from the KS test^a are lower than the 5% significance level, rejecting the hypothesis that the two distributions follow the same continuous distribution, implying that the Skewed Student t distribution may still not be able to model the residuals. On the contrary, the p-values in the KS test^b are very high, supporting that the

Skewed Student t distribution is more appropriate than the standard Student t distribution in modeling the marginal distributions.

The lower panel in Table 2 provides results of the Ljung-Box test and the ARCH LM test. Since the Ljung-Box test assumes the errors being normally distributed, an extra process should be implemented before applying the test. As Smith [1985] and Lai et al. [2009] suggested, the standardized residuals were first transferred into cumulative probabilities by using the Skewed Student t CDF with the estimated parameters; and then the inverse Gaussian CDF function was employed to transfer the observations back to the standard normal variables before applying the Ljung-Box test. As the test statistics show, there is no serial correlation up to 10 lags in the transformed residuals and the squared ones at the 5% significance level. The ARCH effect has been removed from the five equity return series according to the statistics of the ARCH LM test. It is found that the findings based on the estimated parameters and test statistics from other 40 updates are consistent with those discussed above. Therefore, the AR-TGARCH Skewed Student t models should be suitable for modeling the marginal distributions.

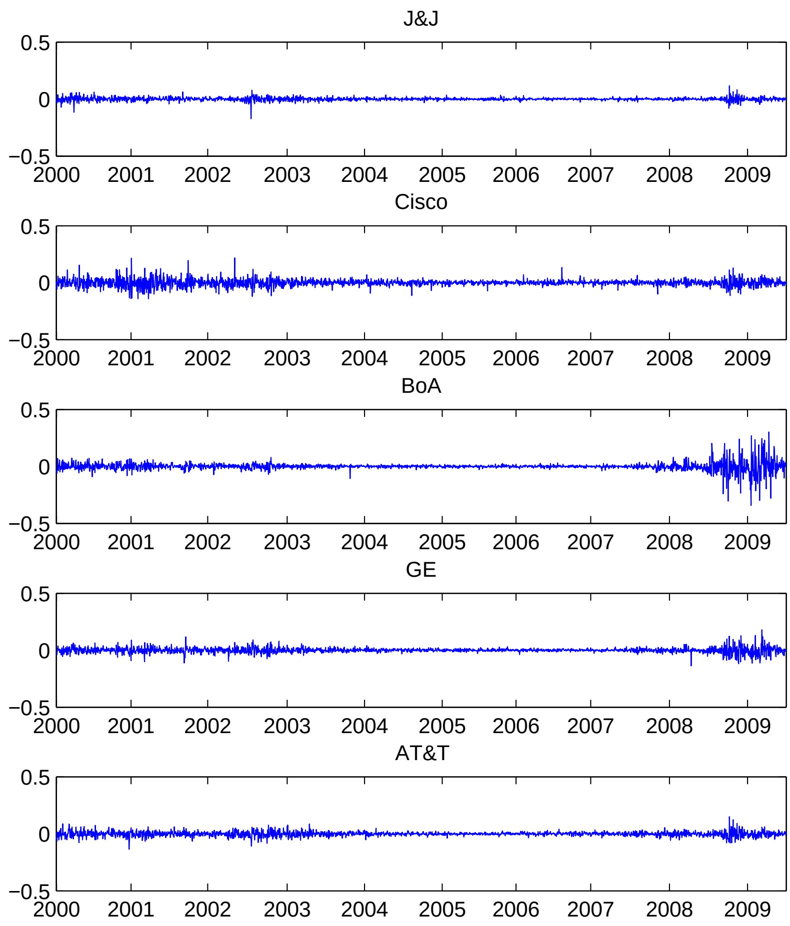


Figure 2: Daily Return of the Five Assets

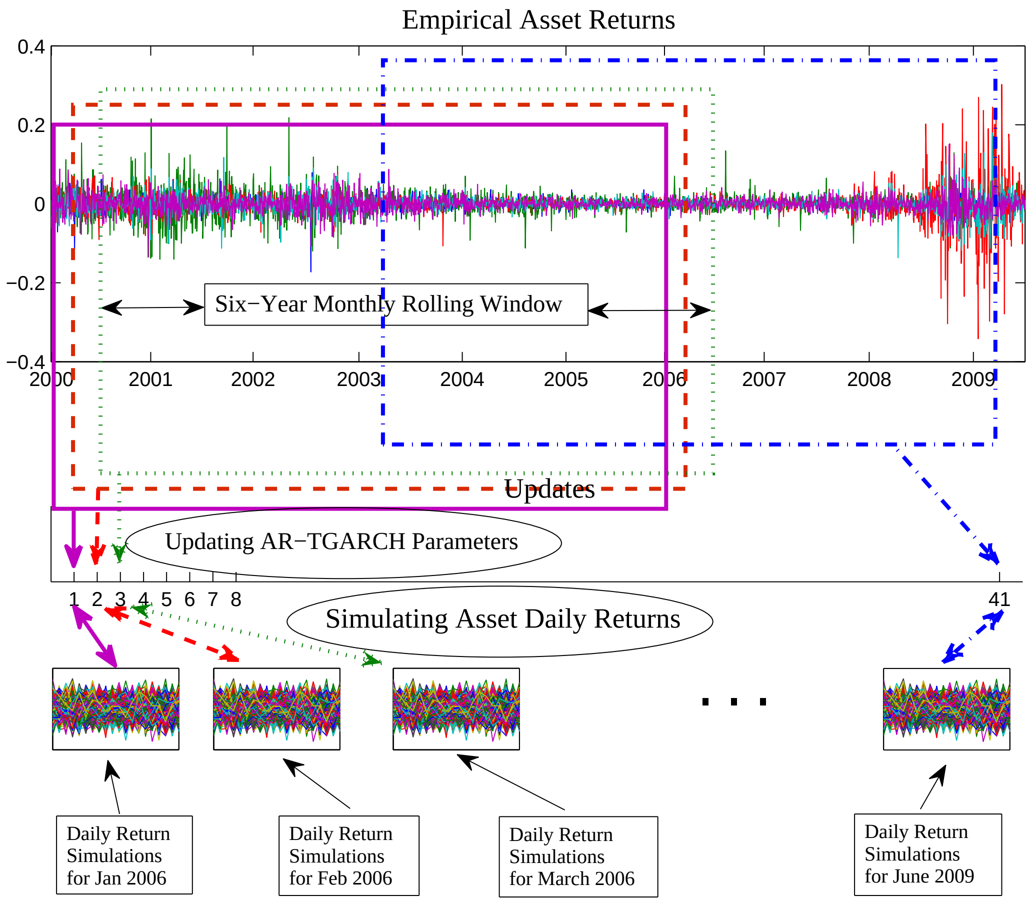
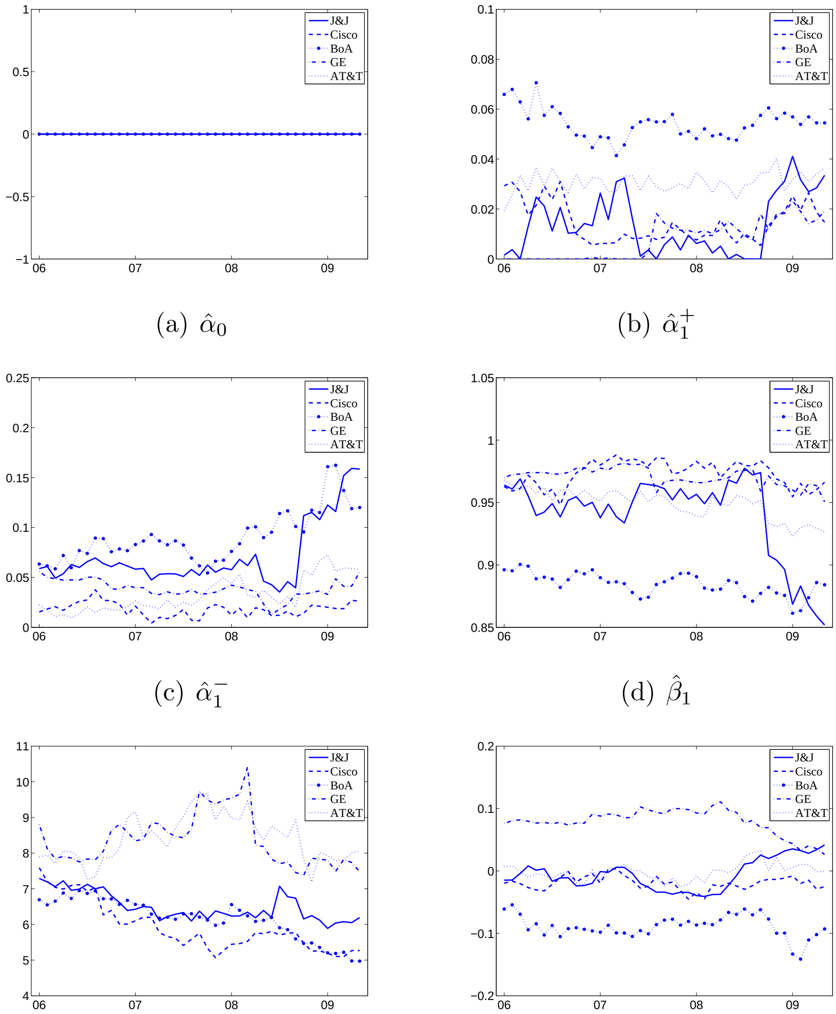


Figure 3: Procedure for Asset Return Simulation from the AR-TGARCH Model



(a) $\hat{\alpha}_0$ (b) $\hat{\alpha}_1^+$

(c) $\hat{\alpha}_1^-$ (d) $\hat{\beta}_1$

(e) $\hat{\eta}$ (f) $\hat{\lambda}$

Figure 4: Estimated Parameters of the TGARCH Models

Table 2: Estimated Parameters of the AR-TGARCH models (the 41st update)

	J&J		Cisco		BoA		GE		AT&T	
AR Process: the parameters and the standard errors.										
	φ_2 -0.0834	(0.0264)	φ_1 -0.0742	(0.0266)	–	–	–	–	φ_7 -0.1200	(0.0264)
	φ_3 -0.0927	(0.0265)	φ_6 -0.0460	(0.0266)	–	–	–	–	–	–
TGARCH: the parameters and the standard errors.										
$\hat{\alpha}_0$	0.0000	(0.0000)	0.0000	(0.0000)	0.0000	(0.0000)	0.0000	(0.0000)	0.0000	(0.0000)
$\hat{\alpha}_1^+$	0.0335	(0.0225)	0.0147	(0.0083)	0.0545	(0.0200)	0.0192	(0.0158)	0.0364	(0.0155)
$\hat{\alpha}_1^-$	0.1585	(0.0432)	0.0259	(0.0130)	0.1200	(0.0376)	0.0557	(0.0291)	0.0578	(0.0225)
$\hat{\beta}_1$	0.8519	(0.0344)	0.9662	(0.0109)	0.8841	(0.0213)	0.9506	(0.0273)	0.9265	(0.0151)
$\hat{\eta}$	6.1926	(0.9976)	5.2685	(0.6737)	4.9709	(0.6606)	7.4804	(1.2851)	8.0605	(1.6466)
$\hat{\lambda}$	0.0416	(0.0356)	-0.0251	(0.0363)	-0.0929	(0.0343)	0.0264	(0.0403)	-0.0008	(0.0575)
Ljung-Box, ARCH LM and KS tests: the statistics and the p-values.										
KS test^a	0.0472	0.0037	0.0586	0.0001	0.0778	0.0000	0.0441	0.0083	0.0445	0.0076
KS test^b	0.0143	0.9337	0.0124	0.9800	0.0259	0.2998	0.0224	0.4789	0.0155	0.8830
Q(1)	0.9260	0.3359	0.4196	0.5172	0.0010	0.9742	2.4560	0.1171	1.3013	0.2540
$Q^2(1)$	0.3273	0.5673	0.5754	0.4481	0.0328	0.8562	0.2903	0.5900	0.2082	0.6482
Q(10)	10.9598	0.3376	11.6855	0.3067	5.8624	0.8267	5.9046	0.8232	16.1547	0.0953
$Q^2(10)$	3.9846	0.9480	10.2861	0.4158	2.4592	0.9915	3.1338	0.9781	5.1397	0.8817
LM(1)	0.4003	0.5269	0.0692	0.7925	0.1095	0.7408	0.0252	0.8738	0.1197	0.7294
$LM^2(1)$	0.4275	0.5132	0.0092	0.9238	0.0012	0.9720	0.0064	0.9363	0.2155	0.6425
LM(10)	4.0265	0.9461	1.4529	0.9991	0.6527	1.0000	1.6126	0.9985	6.4772	0.7737
$LM^2(10)$	4.8374	0.9018	0.2135	1.0000	0.0120	1.0000	0.0661	1.0000	7.0913	0.7168
AIC	-9167.7916		-7110.4721		-7814.2585		-8098.7084		-8013.5092	
LL	4591.8958		3563.2360		3913.1293		4055.3542		4013.7546	

5.3 Estimation of the Copula Models

Table 3 reports the parameter estimates of the two vine pair-copula models of the last update (i.e. the period July 2004 to June 2009). The ordering of the equities (i.e. J&J, Cisco, BoA, GE and AT&T corresponding to the risk factor 1 to 5 in the first chain) is decided based on the ordering of estimated DoF from fitting a bivariate Student copula to each pair of risk factors as Aas et al. [2007] suggested. The subscripts of the pair-copula in the table or the edge label of the vine structure can be found in Figure 1. The upper panel in Table 3 reports the estimated parameters from the maximum likelihood estimation (MLE), and the standard errors extracted from the inverse of the Hessian matrix of the Joe-Clayton pair-copula system. It is found that most of $\hat{\tau_U}$ are greater than $\hat{\tau_L}$, except for the case of J&J and GE conditioning on Cisco and BoA (i.e. $\hat{\tau_L} > \hat{\tau_U}$ in $C_{14|23}$).

Table 3: Estimated Parameters of the Pair-Copula Models (the 41st update)

Joe-Clayton	$\hat{\tau_L}$	$\hat{\tau_U}$		$\hat{\tau_L}$	$\hat{\tau_U}$
C_{12}	0.1070	0.2080	$C_{24\|3}$	0.0130	0.2430
	(0.0014)	(0.0012)		(0.0001)	(0.0010)
C_{23}	0.2140	0.2530	$C_{35\|4}$	0.0170	0.1640
	(0.0015)	(0.0011)		(0.0005)	(0.0073)
C_{34}	0.3550	0.3480	$C_{14\|23}$	0.1010	0.0500
	(0.0011)	(0.0009)		(0.0011)	(0.0015)
C_{45}	0.2510	0.2430	$C_{25\|34}$	0.0030	0.1500
	(0.0013)	(0.0012)		(0.0000)	(0.0010)
$C_{13\|2}$	0.0280	0.1900	$C_{15\|234}$	0.0030	0.0390
	(0.0004)	(0.0008)		(0.0000)	(0.0009)
LL	877.8221				
χ^2 test:	statistic	8.9631	p-value	0.1757	

Student t	$\hat{\rho}$	$\hat{\nu}$		$\hat{\rho}$	$\hat{\nu}$
C_{12}	0.3512	6.3505	$C_{24\|3}$	0.3315	14.1036
	(0.0006)	(1.3870)		(0.0006)	(24.4632)
C_{23}	0.4384	9.1081	$C_{35\|4}$	0.2689	16.1225
	(0.0005)	(5.2193)		(0.0006)	(36.1920)
C_{34}	0.5669	12.0732	$C_{14\|23}$	0.2841	13.3628
	(0.0003)	(15.2316)		(0.0007)	(18.3792)
C_{45}	0.4546	9.4664	$C_{25\|34}$	0.2341	9.9268
	(0.0005)	(6.4654)		(0.0007)	(5.7908)
$C_{13\|2}$	0.2797	20.2949	$C_{15\|234}$	0.1427	38.6106
	(0.0006)	(38.8564)		(0.0007)	(262.528)
LL	976.7462				
χ^2 test:	statistic	11.9468	p-value	0.0632	

(Standard errors of the estimated parameters are provided in parentheses.)

Figure 5 shows the estimated $\hat{\tau_L}$ and $\hat{\tau_U}$ based on a six-year monthly rolling window in the period 3 January 2006 to 30 June 2009. As the first four subplots in chain Υ_1 show, most of the lower rank correlation parameters $\hat{\tau_L}$ increase in the period, whereas the upper rank correlation parameters $\hat{\tau_U}$ remain stably. In addition to the asymmetric dependence structure observed, it is found that the dependence measures of the copulae $\hat{\tau_L}$ and $\hat{\tau_U}$ turn more stable while conditioning on more risk factors (e.g. in the copulae $C_{13|2}$, $C_{14|23}$ and $C_{15|234}$).

The lower panel in Table 3 reports the estimates of $\hat{\rho}$, $\hat{\nu}$, and the standard errors from the Student t pair-copula system of the last update. The estimated DoF parameter $\hat{\nu}$ is reported after $\hat{\rho}$. In practice, when $\hat{\nu}$ is greater than 30, the Student t copula can be approximated by using the Gaussian copula (see Fantazzini [2009]), which does not consider any tail dependence. When $\hat{\nu}$ is smaller than 3, the third and fourth moments of the distribution are not defined. As the table shows, the DoFs of the first four pair-copula in chain Υ_1 are smaller and different from 30 at the 5% significance level, whereas the DoF parameters of the pair-copulae in chains Υ_2, Υ_3 and Υ_4 are not significantly different from 30.

For ease of reading, the $\hat{\nu}$ in the graph has been standardized by dividing by 30 (the DoF which are greater than 30 are replaced by 30 in this case). Figure 6 provides the standardized $\tilde{\hat{\nu}}$ and $\hat{\rho}$ of the examined period based on the six-year monthly rolling window over the period. It is found that the tail dependence measure and the correlation of the copulae are reduced when the pair-copulae are conditional on more risky factors. This finding is consistent with the one observed from the Joe-Clayton pair-copula model.

The p-values from the χ^2 test for the two pair-copula systems are reported in the upper and lower panels of Table 3 after the estimates of the two pair-copula systems from the last update respectively. As the p-values indicate, both the Student t and the Joe-Clayton copula models pass the GoF test at the 5% significance level in the last update. The p-values from the other updates are provided in Table 4. It seems that the Joe-Clayton pair-copula system might not be appropriate in modeling the dependence structure before January 2002, since the p-values of the χ^2 test from the first 24 updates are less than the 5% significance level. In contrast to the Joe-Clayton pair-copula, the Student t pair-copula system passes the test well. According to this statistical test, it seems that the Student t pair-copula system is more suitable than the Joe-Clayton pair-copula for modeling the dependence structure of the five equities.

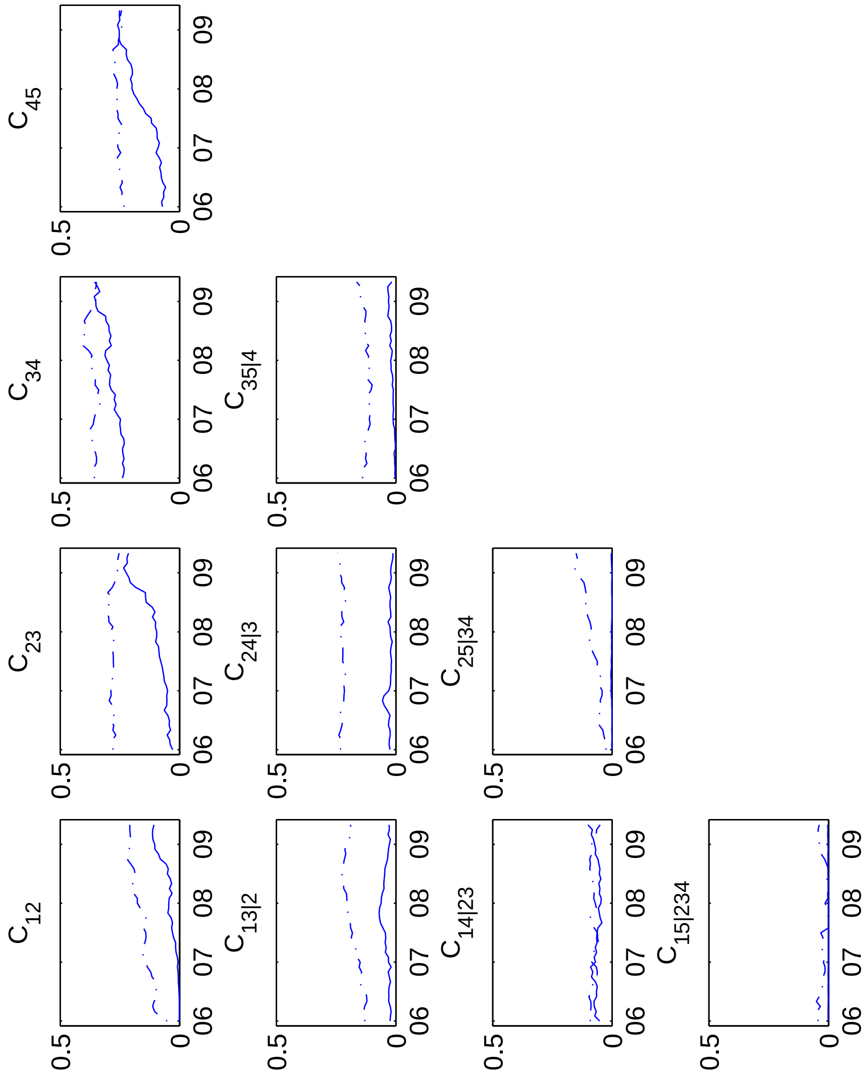


Figure 5: Estimated Parameters of the Joe-Clayton Copula System

($\hat{\tau_L}$ – solid lines and $\hat{\tau_U}$ – dash-dot lines of the Joe-Clayton system. x and y axes represent the time horizon and the rank correlation measures, respectively.)

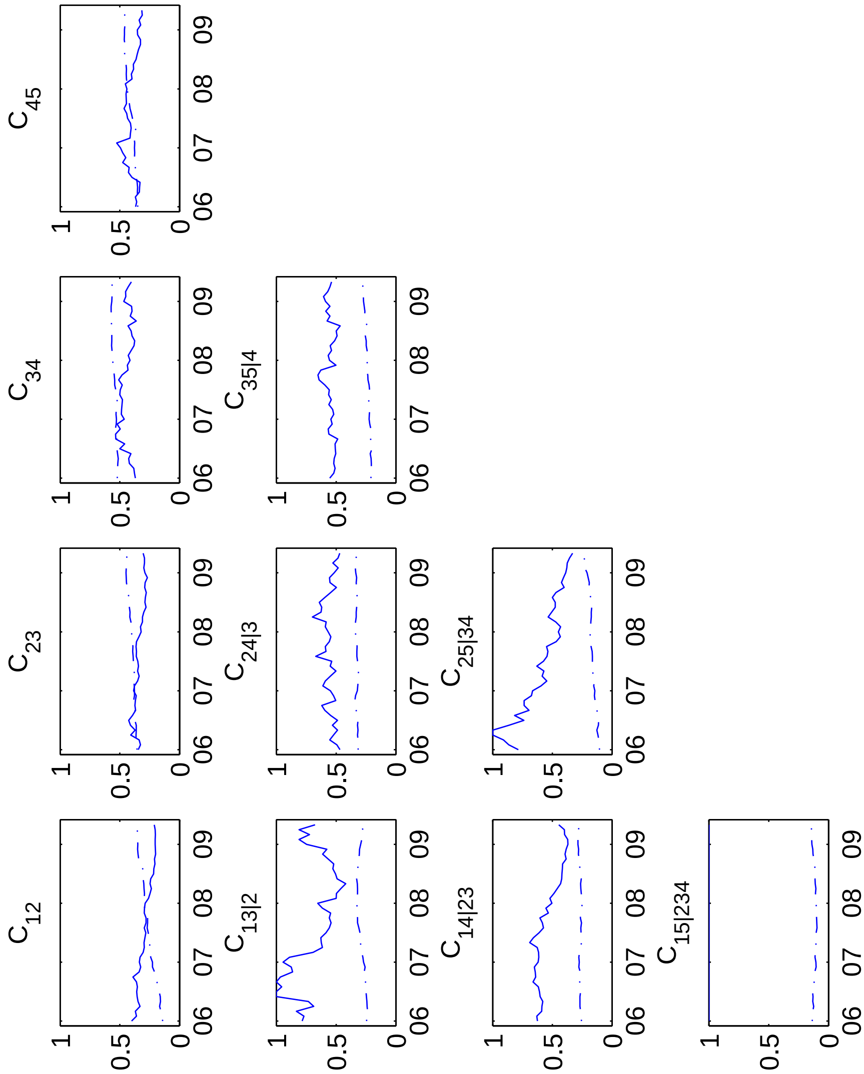


Figure 6: Estimated Parameters of the Student t Copula System

(Standardized $\tilde{\hat{\nu}}$ – solid lines and $\hat{\rho}$ – dash-dot lines of the Student t system. x and y axes represent the time horizon and the copula parameters respectively.)

Table 4: p-Values from the χ^2 Test

Updates	Student t	Joe-Clayton	Updates	Student t	Joe-Clayton
1	0.3857	0.0119	21	0.3640	0.0262
2	0.2876	0.0100	22	0.4766	0.0274
3	0.2059	0.0186	23	0.4675	0.0454
4	0.1571	0.0048	24	0.3668	0.0219
5	0.1605	0.0036	25	0.4736	0.0777
6	0.0998	0.0007	26	0.3453	0.0746
7	0.3818	0.0015	27	0.4175	0.2294
8	0.2231	0.0001	28	0.3565	0.1088
9	0.1735	0.0001	29	0.3153	0.2384
10	0.0889	0.0004	30	0.1545	0.1576
11	0.1746	0.0011	31	0.1668	0.1472
12	0.3257	0.0031	32	0.7488	0.1353
13	0.2757	0.0011	33	0.2777	0.2191
14	0.2726	0.0029	34	0.3611	0.0618
15	0.1865	0.0034	35	0.3376	0.1339
16	0.3796	0.0156	36	0.2824	0.1149
17	0.2625	0.0135	37	0.1950	0.1266
18	0.5222	0.0180	38	0.3994	0.1399
19	0.4417	0.0025	39	0.2132	0.0598
20	0.4777	0.0049	40	0.0497	0.0399

5.4 Copula Selection with Loss Simulation

The loss distribution based on the bootstrapped asset returns can be considered as a benchmark to judge the accuracy of the EWMA model and the pair-copula-GARCH models. The bootstrapped asset returns were directly sampled from historical returns of the period 3 January 2006 to 30 June 2009, thus the simulated loss distribution should be close to the real loss distribution. The simulated loss distributions from the two statistical models were based on the simulated asset returns from the 41 subperiods, which actually were independent of the historical asset returns in the period.

Figure 7 provides the portfolio weights from minimizing the VaR, the ES and the Omega ratio based on the asset returns which are generated by using the two pair-copula-GARCH models, the EWMA model and the bootstrap approach. As the figure shows, the portfolio weights optimized from using the four models agree with each other well when the VaR and the ES measure are minimized, i.e. J&J, Cisco and AT&T are heavily weighted in the four cases. However, the portfolio weights optimized from the Joe-Clayton system and the bootstrap approach, strongly agree with each other while minimizing the Omega ratio as the lower sub-figure in Figure 7 suggested. The portfolio weights from minimizing the Omega ratio can be different from the weights from minimizing VaR and ES, since Omega ratio is developed based on all the moments of return distribution (mean, volatility, skewness, kurtosis and higher moments).

Figure 8 and Figure 9 provide the simulated and the empirically examined (i.e. the bootstrapped) loss distributions in the cases of VaR and ES minimization respectively. As the two figures show, although the gains (i.e. the negative losses) based on the two pair-copula GARCH models match the bootstrapped one closer, the EWMA model is able to provide a closer loss distribution to the bootstrapped one than the pair-copula GARCH models.

However, it is found that the simulated loss distribution based on the Joe-Clayton system better matches the empirically examined loss distribution which is assessed by using the bootstrap approach in overall while minimizing the Omega ratio, as shown in Figure 10. Although the losses from the Joe-Clayton system do not well match the bootstrapped losses in the loss range $[0.5, 2] \times 10^5$, tail events in both of the gains and losses from the two loss distributions agree each other perfectly.

6 Comments and Summary

This paper suggests a pair-copula-TGARCH system to model the joint return distributions of five S&P equities for portfolio risk management. It also discusses

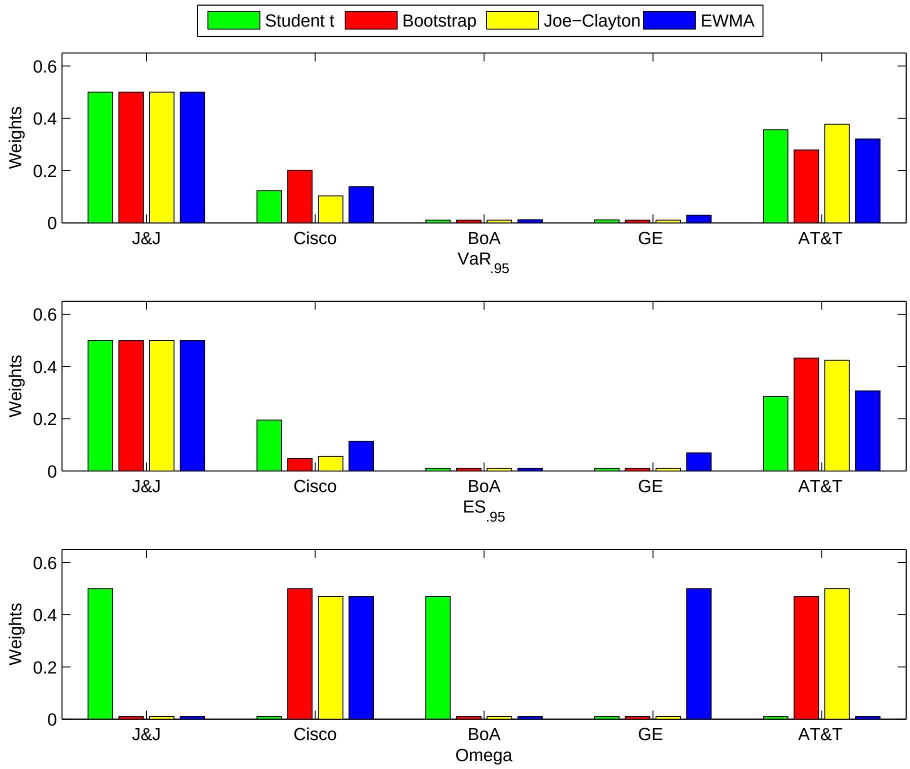


Figure 7: Asset Weights from Downside Risk Minimization

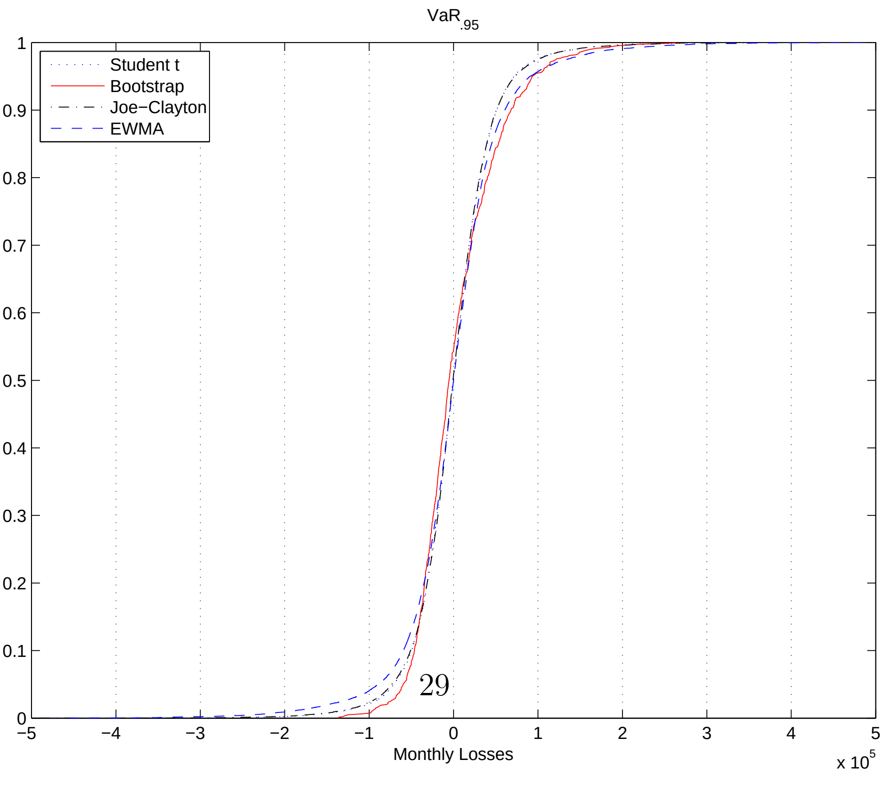


Figure 8: Cumulative Distribution of Monthly Losses for Minimizing $\mathrm{VaR}_{.95}$

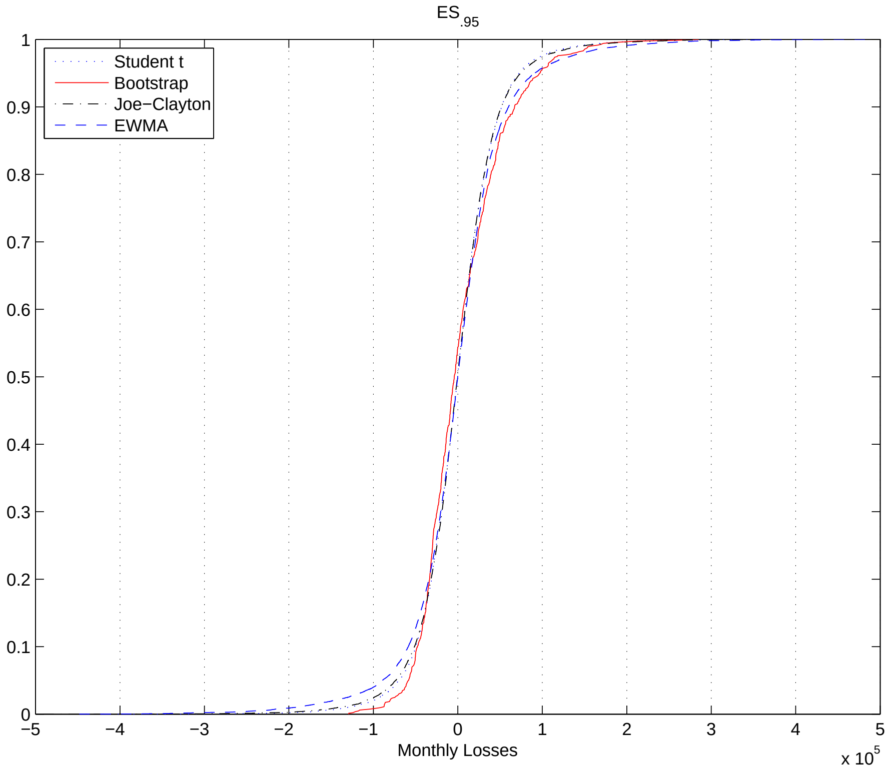


Figure 9: Cumulative Distribution of Monthly Losses for Minimizing ES$_{.95}$

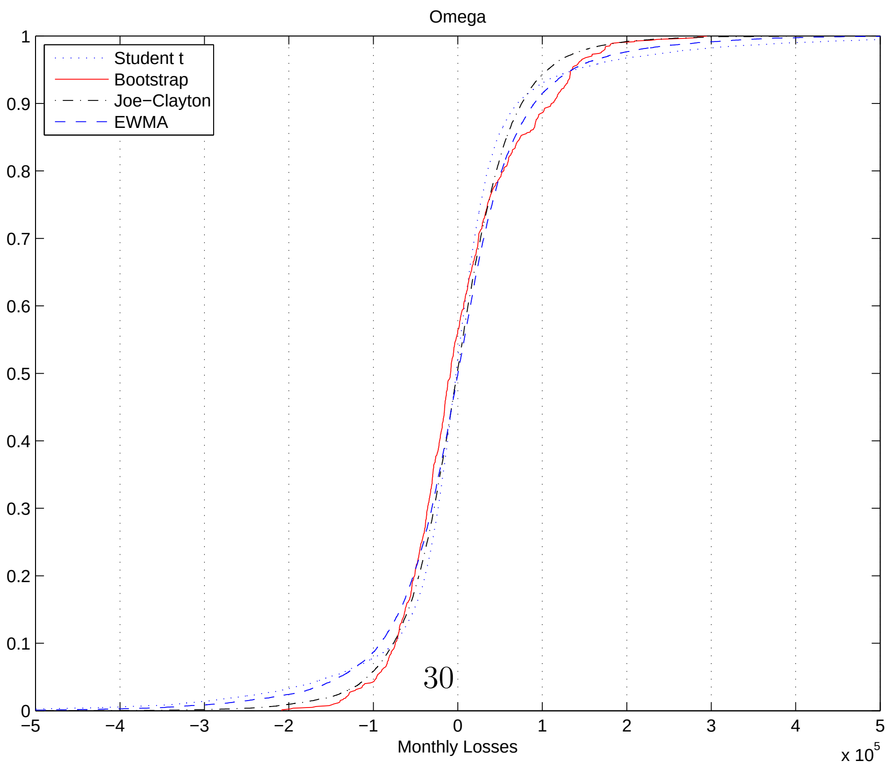


Figure 10: Cumulative Distribution of Monthly Losses for Minimizing Omega

the copula selection problem for the D-vine pair-copula system. The dependence structure of the 5-dimensional case is modeled by using the pair-copula decomposition with the Joe-Clayton copula and the Student t copula. The univariate distributions are modeled by using AR-TGARCH models with the Skewed Student t distribution, which consider the asymmetric effects from past innovations to affect the conditional variance and the stylised facts (skewness and kurtosis). The asset allocation model distributes weights by minimizing three risk indicators, i.e. VaR, ES, and Omega.

As the experimental results suggest, the simple EWMA model is able to provide reliable asset return simulations for the portfolio constructed by minimizing the VaR and ES measures. However, the economic benefit of using the pair-copula GARCH model is revealed by taking the Omega ratio as the risk measure: the portfolio weights and loss distribution from minimizing the Omega ratio based on the Joe-Clayton pair-copula system are more consistent with the empirically examined weights and loss distribution from the bootstrap approach than those from the Student t system and the EWMA model. The advantage of the Joe-Clayton pair-copula system is that it considers an asymmetric dependence structure whereas the Student t copula can only model a symmetric one. Also there is a clear message to financial analysts that selecting copulae for a specified financial problem relies solely on statistical tests may lead to unexpected results. For instance, although the Student t system passes the GoF test better than the Joe-Clayton system, the former provides a less reliable loss distribution when taking the Omega ratio as a risk measure.

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